Proof theory for indexed nested sequents

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Tableaux'17
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Sequent calculus for modal logic

Sequent calculus for modal logic

Formulas: $A ::= a \mid \bar{a} \mid A \wedge A \mid A \vee A$

$A \supset B \equiv (\neg A) \vee B$

Logic K**:** Classical Propositional Logic

Sequent calculus for modal logic

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Logic K**:** Classical Propositional Logic

Sequent system:

$$id \frac{}{\Gamma, \bar{a}, a} \qquad \wedge \frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad \vee \frac{\Gamma, A, B}{\Gamma, A \vee B}$$

Sequent calculus for modal logic

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Logic K**:** Classical Propositional Logic

$+ \quad \mathsf{k} : \Box(A \supset B) \supset (\Box A \supset \Box B) \qquad +$ necessitation: $\dfrac{A}{\Box A}$

Semantics: Relational models (W, R) (Kripke 1963)

Sequent system: (Onishi and Matsumoto 1957)

$$id \; \frac{}{\Gamma, \bar{a}, a} \qquad \wedge \; \frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad \vee \; \frac{\Gamma, A, B}{\Gamma, A \vee B} \qquad k \; \frac{\Gamma, A}{\Diamond\Gamma, \Box A}$$

Extensions

Scott-Lemmon axioms: for a tuple (k, l, m, n) of natural numbers,

$$\mathrm{g}_{\mathrm{klmn}}: (\Diamond^k \Box^l A \supset \Box^m \Diamond^n A) \wedge (\Diamond^m \Box^n A \supset \Box^k \Diamond^l A)$$

where $\Box^m$ stands for m boxes and $\Diamond^n$ for n diamonds.

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Frame property: (Scott and Lemmon 1977)
for all $w, u, v \in W$ with $wR^k u$ and $wR^m v$,
there is a $z \in W$ such that $uR^l z$ and $vR^n z$.

Nested sequents

Nested sequents generalise sequents from a multiset of formulas

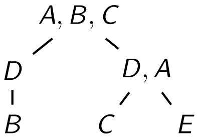
Sequent:

$$A, B, C$$

Nested sequents

Nested sequents generalise sequents from a multiset of formulas to a tree of multisets of formulas.

Nested sequent:



(Brünnler, 2009), (Poggiolesi, 2009)

Nested sequents

In the sequent term, brackets indicate the parent-child relation in the tree

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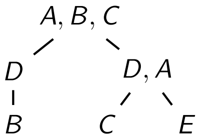


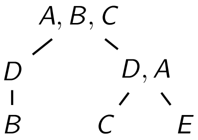
$$\Gamma = A, B, C, [D, [B]], [D, A, [C], [E]]$$

(Brünnler, 2009), (Poggiolesi, 2009)

Nested sequents

In the sequent term, brackets indicate the parent-child relation in the tree and can be interpreted as the modal $\Box$.

Nested sequent:



$$\Gamma = A, B, C, [D, [B]], [D, A, [C], [E]]$$

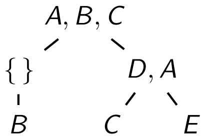
$$A \lor B \lor C \lor \Box(D \lor \Box B) \lor \Box(D \lor A \lor \Box C \lor \Box E)$$

(Brünnler, 2009), (Poggiolesi, 2009)

Nested sequents

A context is obtained by removing a formula and replacing it by a hole

Sequent context:



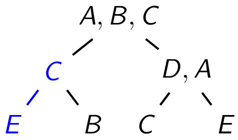
$$\Gamma\{\ \} = A, B, C, [\{\ \}, [B]], [D, A, [C], [E]]$$

(Brünnler, 2009), (Poggiolesi, 2009)

Nested sequents

A context is obtained by removing a formula and replacing it by a hole that can then be filled by another nested sequent.

Sequent context:



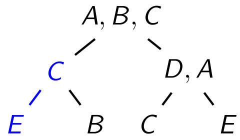
$$\Gamma\{C, [E]\} = A, B, C, [C, [E], [B]], [D, A, [C], [E]]$$

(Brünnler, 2009), (Poggiolesi, 2009)

Nested sequents

This allows us to build rules than can be applied at any depth in the tree.

Sequent context:



$$\Gamma\{C, [E]\} = A, B, C, [C, [E], [B]], [D, A, [C], [E]]$$

(Brünnler, 2009), (Poggiolesi, 2009)

Nested sequents

Sequent-like rules:

$$\wedge\,\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \quad \rightsquigarrow \quad \wedge\,\frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \qquad\qquad \vee\,\frac{\Gamma, A, B}{\Gamma, A \vee B} \quad \rightsquigarrow \quad \vee\,\frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}}$$

Nested sequents

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Nested rules:

$$\Box\,\frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \Diamond\,\frac{\Gamma\{[A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}$$

$$k\,\frac{\Delta, A}{\Diamond\Delta, \Box A} \quad \rightsquigarrow \quad \Diamond\,\frac{\begin{array}{c}\Gamma\{[\Diamond B_1, \ldots, \Diamond B_n, A]\} \\ \Diamond\big\Vert \\ \Gamma\{\Diamond B_1, \ldots, \Diamond B_{n-1}, [\Diamond B_n, A]\}\end{array}}{\Box\,\dfrac{\Gamma\{\Diamond B_1, \ldots, \Diamond B_n, [A]\}}{\Gamma\{\Diamond B_1, \ldots, \Diamond B_n, \Box A\}}}$$

Nested sequents

Nested Sequent: $\Gamma ::= A_1, \ldots, A_m, [\Gamma_1], \ldots, [\Gamma_n]$

Corresponding formula: $\mathrm{fm}(\Gamma) = A_1 \vee \ldots \vee A_m \vee \Box \mathrm{fm}(\Gamma_1) \vee \ldots \vee \Box \mathrm{fm}(\Gamma_n)$

Sequent context: $\Gamma\{\ \}\{\ \}\{\ \} = A, [\{\ \}], [B, \{\ \}, [\{\ \}]]$

System nK**:**

$$\mathsf{id}\ \frac{}{\Gamma\{a, \bar{a}\}} \quad \vee\ \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}} \quad \wedge\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \quad \Box\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \quad \Diamond\ \frac{\Gamma\{\Diamond A, [A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}$$

Theorem: *System* nK *is sound and complete for the logic* K*.*

(Brünnler, 2009), (Poggiolesi, 2009)

Indexed nested sequents

Indexed Nested Sequent: $\Gamma ::= A_1, \ldots, A_m, [^{w_1}\Gamma_1], \ldots, [^{w_n}\Gamma_n]$

No corresponding formula in the general case

Indexed context: $\Gamma\{^2\ \}\{^1\ \}\{^2\ \} = A, [^2\{\ \}], [^1 B, \{\ \}, [^2\{\ \}]]$

System inK**:**

$$\text{id}\ \frac{}{\Gamma\{a, \bar{a}\}} \quad \vee\ \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}} \quad \wedge\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \quad \Box\ \frac{\Gamma\{[^v A]\}}{\Gamma\{\Box A\}} \quad \Diamond\ \frac{\Gamma\{\Diamond A, [^u A, \Delta]\}}{\Gamma\{\Diamond A, [^u \Delta]\}}$$

(Fitting, 2015)

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Indexed context: $\Gamma\{^2 C\}\{^1\ \}\{^2\ \} = A, [^2 C], [^1 B, \{\ \}, [^2 \{\ \}]]$

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Extensions

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Corresponding rule: (Fitting 2015)

$$\boxtimes_{\mathrm{g_{klmn}}} \frac{\Gamma\{^{u_0}[^{u_1}\Delta_1, \ldots [^{u_k}\Delta_k, [^{v_1} \ldots [^{v_l} \;\;] \ldots]] \ldots], [^{w_1}\Sigma_1, \ldots [^{w_m}\Sigma_m, [^{x_1} \ldots [^{x_n} \;\;] \ldots]] \ldots]\}}{\Gamma\{^{u_0}[^{u_1}\Delta_1, \ldots [^{u_k}\Delta_k] \ldots], [^{w_1}\Sigma_1, \ldots [^{w_m}\Sigma_m] \ldots]\}}$$

$v_1 \ldots v_k$ and $x_1 \ldots x_n$ are fresh indexes which are pairwise distinct, except

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Theorem:
System inK $+ \boxtimes_{\mathrm{g_{klmn}}}$ *is sound and complete for the logic* K $+ \mathrm{g_{klmn}}$.

Intuitionistic modal logics

Formulas: $A ::= a \mid A \wedge A \mid A \vee A \mid \bot \mid A \supset A$

Logic IK**:** Intuitionistic Propositional Logic

Intuitionistic modal logics

Formulas: $A ::= a \mid A \wedge A \mid A \vee A \mid \bot \mid A \supset A \mid \Box A \mid \Diamond A$

Logic IK**:** Intuitionistic Propositional Logic

$$+ \quad \begin{array}{ll} \mathsf{k1}: & \Box(A \supset B) \supset (\Box A \supset \Box B) \\ \mathsf{k2}: & \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) \\ \mathsf{k3}: & \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B) \\ \mathsf{k4}: & (\Diamond A \supset \Box B) \supset \Box(A \supset B) \\ \mathsf{k5}: & \Diamond \bot \supset \bot \end{array} \quad + \quad \text{necessitation: } \frac{A}{\Box A}$$

(Plotkin and Sterling 1986)

Intuitionistic modal logics

Formulas: $A ::= a \mid A \wedge A \mid A \vee A \mid \bot \mid A \supset A \mid \Box A \mid \Diamond A$

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(Plotkin and Sterling 1986)

Kripke semantics: (Bi)relational structures $(W, R, \leq)$ (Fischer-Servi 1984)

- a non-empty set W of *worlds*;
- a binary relation $R \subseteq W \times W$;
- and a preorder $\leq$ on W, such that:

(F1) For all worlds u, v, v', if uRv and $v \leq v'$, then there exists a u' such that $u \leq u'$ and $u'Rv'$.

(F2) For all worlds u', u, v, if $u \leq u'$ and uRv, then there exists a v' such that $u'Rv'$ and $v \leq v'$.

Intuitionistic modal logics

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Sequent system:

$$\mathsf{k}_{\Box}\ \frac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A} \qquad \mathsf{k}_{\Diamond}\ \frac{\Gamma, A \Rightarrow B}{\Box\Gamma, \Diamond A \Rightarrow \Diamond B}$$

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Problem? k3, k4 and k5 are not derivable.

- not a problem for modal type theory...

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Problem? k3, k4 and k5 are not derivable.

- not a problem for modal type theory...

Labelled sequent system: (Simpson 1994)

$$\Box_L \ \frac{xRy, \Gamma, x\colon \Box A, y\colon A \Rightarrow z\colon B}{xRy, \Gamma, x\colon \Box A \Rightarrow z\colon B} \qquad \Box_R \ \frac{xRy, \Gamma \Rightarrow y\colon A}{\Gamma \Rightarrow x\colon \Box A} \ y \text{ is fresh}$$

$$\Diamond_L \ \frac{xRy, \Gamma, y\colon A \Rightarrow z\colon B}{\Gamma, x\colon \Diamond A \Rightarrow z\colon B} \ y \text{ is fresh} \qquad \Diamond_R \ \frac{xRy, \Gamma \Rightarrow y\colon A}{xRy, \Gamma \Rightarrow x\colon \Diamond A}$$

Intuitionistic modal logics

Sequent system:

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Problem? k3, k4 and k5 are not derivable.

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Labelled sequent system: (Simpson 1994)

$$\Box_{\mathsf{L}}\ \frac{xRy, \Gamma, x : \Box A, y : A \Rightarrow z : B}{xRy, \Gamma, x : \Box A \Rightarrow z : B} \qquad \Box_{\mathsf{R}}\ \frac{xRy, \Gamma \Rightarrow y : A}{\Gamma \Rightarrow x : \Box A}\ y \text{ is fresh}$$

$$\Diamond_{\mathsf{L}}\ \frac{xRy, \Gamma, y : A \Rightarrow z : B}{\Gamma, x : \Diamond A \Rightarrow z : B}\ y \text{ is fresh} \qquad \Diamond_{\mathsf{R}}\ \frac{xRy, \Gamma \Rightarrow y : A}{xRy, \Gamma \Rightarrow x : \Diamond A}$$

Controversy: (Restall 2006)

1. Multiplicity
2. Subformula property

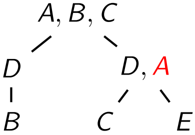
Nested sequents for intuitionistic modal logic

Nested sequents generalise sequents from a multiset of formulas with one formula distinguished on the right.

$$A, B \Rightarrow C$$

Nested sequents for intuitionistic modal logic

Nested sequents generalise sequents from a multiset of formulas *to a tree of multisets of formulas* with one formula distinguished in the whole tree.



(Straßburger 2013)

Indexed nested sequents for intuitionistic modal logic

System inIK:

$$\text{id}\ \frac{}{\Gamma\{a, a\}} \qquad \wedge_L\ \frac{\Gamma\{A, B\}}{\Gamma\{A \wedge B\}} \qquad \wedge_R\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}}$$

$$\bot_L\ \frac{}{\Gamma\{\bot\}} \qquad \supset_L\ \frac{\Gamma^\star\{A \supset B, A\} \quad \Gamma\{B\}}{\Gamma\{A \supset B\}} \qquad \supset_R\ \frac{\Gamma\{A, B\}}{\Gamma\{A \supset B\}}$$

$$\vee_L\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \vee B\}} \qquad \vee_{R1}\ \frac{\Gamma\{A\}}{\Gamma\{A \vee B\}} \qquad \vee_{R2}\ \frac{\Gamma\{B\}}{\Gamma\{A \vee B\}}$$

$$\Box_L\ \frac{\Gamma\{\Box A, [^w A, \Delta]\}}{\Gamma\{\Box A, [^w \Delta]\}} \qquad \Box_R\ \frac{\Gamma\{[^v A]\}}{\Gamma\{\Box A\}} \qquad \Diamond_L\ \frac{\Gamma\{[^v A]\}}{\Gamma\{\Diamond A\}} \qquad \Diamond_R\ \frac{\Gamma\{[^w A, \Delta]\}}{\Gamma\{\Diamond A, [^w \Delta]\}}$$

. . . .

$$\text{tp}\ \frac{\Gamma\{^w \varnothing\}\{^w A\}}{\Gamma\{^w A\}\{^w \varnothing\}} \qquad \text{bc}_1\ \frac{\Gamma_1\{^w[^u \Gamma_2]\}\{[^u \varnothing]\}}{\Gamma_1\{^w[^u \Gamma_2]\}\{^w \varnothing\}} \qquad \text{bc}_2\ \frac{\Gamma_1\{^w[^u \Gamma_2\{^w[^u \varnothing]\}]\}}{\Gamma_1\{^w[^u \Gamma_2\{^w \varnothing\}]\}}$$

Extensions

G a set of Scott-Lemmon axioms and $\boxtimes_{\mathsf{G}}$ the corresponding set of rules.

Cut-elimination: If Γ is provable in $\mathsf{inIK} + \boxtimes_{\mathsf{G}} + \mathsf{cut}$, where

$$\mathsf{cut}\,\frac{\Gamma\{A\} \quad \Gamma\{A\}}{\Gamma\{\,\}}$$

then Γ is provable in $\mathsf{inIK} + \boxtimes_{\mathsf{G}}$.

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$$\mathsf{cut}\ \frac{\Gamma\{A\} \quad \Gamma\{A\}}{\Gamma\{\ \}} \qquad \text{then } \Gamma \text{ is provable in } \mathsf{inIK} + \boxtimes_{\mathsf{G}}.$$

Completeness: If A is provable in the Hilbert system $\mathsf{IK} + \mathsf{G}$, then A is provable in the indexed nested sequent system $\mathsf{inIK} + \boxtimes_{\mathsf{G}}$.

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Cut-elimination: If Γ is provable in $\mathsf{inIK} + \boxtimes_{\mathsf{G}} + \mathsf{cut}$, where

$$\mathsf{cut}\ \dfrac{\Gamma\{A\} \quad \Gamma\{A\}}{\Gamma\{\ \}}$$

then Γ is provable in $\mathsf{inIK} + \boxtimes_{\mathsf{G}}$.

Completeness: If A is provable in the Hilbert system $\mathsf{IK} + \mathsf{G}$, then A is provable in the indexed nested sequent system $\mathsf{inIK} + \boxtimes_{\mathsf{G}}$.

$$
\mathsf{\supset_R}\ \dfrac{
\mathsf{\Diamond_L}, \mathsf{\Box_R}\ \dfrac{
\boxtimes_{\mathsf{g}_{klmn}}\ \dfrac{
\mathsf{\Box_L}, \mathsf{\Diamond_R}\ \dfrac{
\mathsf{tp}\ \dfrac{
\mathsf{id}\ \dfrac{}{[^{u_1}\dots[^{u_k}\Box^l p, [^{v_1}\Box^{l-1}p, \dots [^{v_{l-1}}\Box p, [^{v_l}\ ]]\dots]]\dots],\ [^{w_1}\dots[^{w_m}[^{x_1}\dots[^{x_{n-1}}[^{x_n} p, p]]\dots]]\dots]}
}{[^{u_1}\dots[^{u_k}\Box^l p, [^{v_1}\Box^{l-1}p, \dots [^{v_{l-1}}\Box p, [^{v_l} p]]\dots]]\dots],\ [^{w_1}\dots[^{w_m}[^{x_1}\dots[^{x_{n-1}}[^{x_n} p]]\dots]]\dots]}\ c_l = d_n
}{[^{u_1}\dots[^{u_k}\Box^l p, [^{v_1}\dots[^{v_l}\ ]\dots]]\dots],\ [^{w_1}\dots[^{w_m}\Diamond^n p, [^{x_1}\dots[^{x_n}\ ]\dots]]\dots]}
}{[^{u_1}\dots[^{u_k}\Box^l p]\dots],\ [^{w_1}\dots[^{w_m}\Diamond^n p]\dots]}
}{\Diamond^k\Box^l p, \Box^m\Diamond^n p}
}{\Diamond^k\Box^l p \supset \Box^m\Diamond^n p}
$$

Extensions

Counter-example to soundness: (Simpson 1994)
The formula:

$$F = (\Diamond(\Box(a \vee b) \wedge \Diamond a) \wedge \Diamond(\Box(a \vee b) \wedge \Diamond b)) \supset \Diamond(\Diamond a \wedge \Diamond b)$$

is derivable in inIK $+ \boxtimes_{\mathrm{g}_{1111}}$, but is not a theorem of IK $+ \mathrm{g}_{1111}$.

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- what about birelational models?

Extensions

Graph-consistency: (Simpson 1994)
A intuitionistic model $\mathcal{M}$ is called graph-consistent if for any sequent Γ, given any homomorphism $h\colon \Gamma \mapsto \mathcal{M}$, any index w appearing in Γ, and any $w' \geq h(w)$, there exists another homomorphism $h'\colon \Gamma \mapsto \mathcal{M}$ such that $h' \geq h$ and $h'(w) = w'$.

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Soundness wrt. graph-consistent models:
If A is provable in $\mathsf{inIK} + \boxtimes_{\mathsf{G}}$ then it is valid in every graph-consistent model satisfying the corresponding Scott-Lemmon frame properties.

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Completeness? Is there a certain set of Scott-Lemmon axioms G such that there exists a formula that is valid in every corresponding graph-consistent models, but that is not a theorem of $\mathsf{inIK} + \boxtimes_{\mathsf{G}}$?

Conclusions

Study of some proof-theoretical properties of indexed nested sequents:

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Separation of classes/logics?
nested sequents $\subset$ indexed nested sequents $\subset$ labelled sequents
(Goré and Ramanayake 2012) (Ramanayake 2016)