Nested sequents for modal logics and beyond

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July 7, 2018

This presentation was made possible by grant NPRP 097-988-1-178, from the Qatar National Research Fund (a member of the Qatar Foundation). The statements made herein are solely the responsibility of the author.
Survey talk on nested sequents

What are nested sequents?

What can they achieve?

1. for logics without a sequent system
   - intuitionistic modal logic IK
2. for sequent systems without a cut-free version
   - classical modal logic S5
3. for cut-free systems without a syntactic cut-elimination procedure
   - modal fixed-point logic

Where will they take you?
What are nested sequents?
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What can they achieve?
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   ▶ intuitionistic modal logic IK
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   - modal fixed-point logic

Where will they take you?
What are nested sequents?
Syntactical term encoding of semantical (tree) structure

[Brünnler, 2009] [Poggiolesi, 2009]
Syntactical term encoding of semantical (tree) structure

Nested sequents:

\[
\left[ \begin{array}{c}
\wedge_0 \\
\end{array} \right]
\]

[Brünnler, 2009] [Poggiolesi, 2009]
From semantics to syntax

**Syntactical** term encoding of **semantical** (tree) structure

Nested sequents:

\[ [0 \bar{p}, s, \ldots] \]

[Brünnler, 2009] [Poggiolesi, 2009]
Syntactical term encoding of semantical (tree) structure

Nested sequents:

\[
\begin{array}{c}
\mbox{0} \bar{\rho}, s, \ldots, \mbox{1} \\
\end{array}
\]
From semantics to syntax

**Syntactical** term encoding of **semantical** (tree) structure

Nested sequents:

\[
\begin{align*}
\{ & 0 \bar{p}, s, \ldots, 1 p, \bar{s}, \ldots \} \\
\end{align*}
\]

[Brünnler, 2009] [Poggiolesi, 2009]
**From semantics to syntax**

**Syntactical** term encoding of **semantical** (tree) structure

Nested sequents:

\[
\begin{bmatrix}
0 \bar{p}, s, \ldots, 1 p, \bar{s}, \ldots, 2 \\
\end{bmatrix}
\]

[Brünnler, 2009] [Poggiolesi, 2009]
From semantics to syntax

**Syntactical** term encoding of **semantical** (tree) structure

Nested sequents:

\[
\begin{array}{c}
[0\bar{p}, s, \ldots, [1p, \bar{s}, \ldots], [2p, s, \ldots]
\end{array}
\]

[Brünner, 2009] [Poggiolesi, 2009]
From semantics to syntax

**Syntactical** term encoding of semantical (tree) structure

Nested sequents:

\[
\left[ 0 \bar{p}, s, \ldots, [ 1 p, \bar{s}, \ldots ] , [ 2 p, s, \ldots, [ 3 \ldots, [ 5 \ldots ], [ 7 \ldots ] ], [ 4 \ldots ], [ 5 \ldots ]] \right]
\]

[Brünnler, 2009] [Poggiolesi, 2009]
What can they achieve?
Intuitionistic modal logic IK is obtained from intuitionistic propositional logic

- by adding the *necessitation rule*: □A is a theorem if A is a theorem;
- and the following five variants of the k axiom.

\[
\begin{align*}
k_1: & \quad \square(A \supset B) \supset (\square A \supset \square B) \\
k_2: & \quad \square(A \supset B) \supset (\Diamond A \supset \Diamond B) \\
k_3: & \quad \Diamond(A \lor B) \supset (\Diamond A \lor \Diamond B) \\
k_4: & \quad (\Diamond A \supset \square B) \supset \square(A \supset B) \\
k_5: & \quad \Diamond \bot \supset \bot
\end{align*}
\]
Intuitionistic modal logic IK is obtained from intuitionistic propositional logic

- by adding the *necessitation rule*: □A is a theorem if A is a theorem;
- and the following five variants of the k axiom.

\[
\begin{align*}
\text{k}_1: & \quad \Box(A \supset B) \supset (\Box A \supset \Box B) \\
\text{k}_2: & \quad \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) \\
\text{k}_3: & \quad \Diamond(A \lor B) \supset (\Diamond A \lor \Diamond B) \\
\text{k}_4: & \quad (\Diamond A \supset \Box B) \supset \Box(A \supset B) \\
\text{k}_5: & \quad \Diamond \bot \supset \bot
\end{align*}
\]

**Sequent system:**

\[
\begin{align*}
\Box_k & \quad \Lambda \Rightarrow A \\
\Box_k & \quad \Box \Lambda \Rightarrow \Box A \\
\Diamond_k & \quad \Lambda, A \Rightarrow B \\
\Diamond_k & \quad \Box \Lambda, \Diamond A \Rightarrow \Diamond B
\end{align*}
\]
Intuitionistic modal logic IK is obtained from intuitionistic propositional logic by adding the \textit{necessitation rule}: $\Box A$ is a theorem if $A$ is a theorem; and the following five variants of the k axiom.

\begin{align*}
k_1: & \quad \Box (A \supset B) \supset (\Box A \supset \Box B) \\
k_2: & \quad \Box (A \supset B) \supset (\Diamond A \supset \Diamond B) \\
k_3: & \quad (A \lor B) \supset (\Diamond A \lor \Diamond B) \\
k_4: & \quad (\Diamond A \supset \Box B) \supset \Box (A \supset B) \\
k_5: & \quad \Diamond \bot \supset \bot
\end{align*}

\textbf{Sequent system:}

\begin{align*}
\vdash & \quad \Box \Lambda \Rightarrow A \\
\vdash & \quad \Diamond \Lambda, \Box \Rightarrow \Diamond A \\
\vdash & \quad \Box \Lambda, A \Rightarrow B \\
\vdash & \quad \Box \Lambda, \Diamond A \Rightarrow \Diamond B \\
\vdash & \quad \Diamond \bot \Rightarrow \bot
\end{align*}

\textbf{Theorem:} \( \text{LJ}_p + \Box^o_k + \Diamond^o_k \) is sound and complete for \( \text{IK} - \{k_3, k_4, k_5\} \).
Intuitionistic modal logic IK is obtained from intuitionistic propositional logic

- by adding the necessity rule: □A is a theorem if A is a theorem;
- and the following five variants of the k axiom.

\[
\begin{align*}
\text{k}_1 & : \quad \square(A \supset B) \supset (\square A \supset \square B) \\
\text{k}_2 & : \quad \square(A \supset B) \supset (\Diamond A \supset \Diamond B) \\
\text{k}_3 & : \quad (A \lor B) \supset (\Diamond A \lor \Diamond B) \\
\text{k}_4 & : \quad (\Diamond A \supset \square B) \supset \square(A \supset B) \\
\text{k}_5 & : \quad \Diamond \bot \supset \bot
\end{align*}
\]

Nested sequent system:

\[
\begin{align*}
\text{n}_R & \quad \text{n}_k \\
\frac{\Delta_1\{\square A, [A, \Delta_2]\}}{\Lambda_1\{[\Lambda_2], \Diamond A\}} & \quad \frac{\Pi\{\Diamond A\}}{\Pi\{\Lambda_2\}} \\
\frac{\Lambda_1\{[\Lambda_2, A]\}}{\Lambda_1\{[\Lambda_2, \Diamond A]\}} & \quad \frac{\Lambda\{[A]\}}{\Lambda\{\square A\}} \quad \frac{\Delta\{\square A, [A, \Delta_2]\}}{\Delta_1\{\square A, [\Delta_2]\}}
\end{align*}
\]
Intuitionistic modal logic IK is obtained from intuitionistic propositional logic

- by adding the *necessitation rule*: \( \Box A \) is a theorem if \( A \) is a theorem;
- and the following five variants of the \( k \) axiom.

\[ \begin{align*}
k_1 : & \quad \Box (A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B) \\
k_2 : & \quad (\Box (A \rightarrow B)) \rightarrow (\Box A \rightarrow \Box B) \\
k_3 : & \quad \Diamond (A \lor B) \rightarrow (\Diamond A \lor \Diamond B) \\
k_4 : & \quad (\Diamond A \lor \Box B) \rightarrow \Box (A \lor B) \\
k_5 : & \quad \uparrow \perp \rightarrow \downarrow \perp
\end{align*} \]

**Nested sequent system:**

\[ \begin{align*}
& \frac{\Box^n \Lambda_1 \{[\Lambda_2, A]\} \quad L^n \Pi \{\Diamond A\} \quad R^n \Lambda \{[A]\} \quad L^n \Delta_1 \{\Box A, [A, \Delta_2]\}}{\Box^n \Lambda_1 \{[\Lambda_2, \Diamond A]\}}
\end{align*} \]

**Theorem:** nIK is sound and complete for IK.  
[Strässburger, 2013]
Intuitionistic modal logic $IK$ is obtained from intuitionistic propositional logic

- by adding the *necessitation rule*: $\Box A$ is a theorem if $A$ is a theorem;
- and the following five variants of the $k$ axiom.

\[
\begin{align*}
    k_1: & \quad \Box(A \supset B) \supset (\Box A \supset \Box B) \\
    k_2: & \quad \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) \\
    k_3: & \quad (A \lor B) \supset (\Diamond A \lor \Diamond B) \\
    k_4: & \quad (\Diamond A \supset \Box B) \supset \Box(A \supset B) \\
    k_5: & \quad \bot \supset \bot
\end{align*}
\]

*Nested sequent system:*

\[
\begin{align*}
    & \Diamond^n_{Rk} \frac{\Lambda_1\{[\Lambda_2, A]\}}{\Lambda_1\{[\Lambda_2], \Diamond A\}} \\
    & \Pi^n_L \frac{\Pi\{\Diamond A\}}{\Lambda\{[A]\}} \\
    & \Box^n_R \frac{\Lambda\{[A]\}}{\Lambda\{\Box A\}} \\
    & \Box^n_{Lk} \frac{\Delta_1\{\Box A, [A, \Delta_2]\}}{\Delta_1\{\Box A, [\Delta_2]\}}
\end{align*}
\]

**Theorem:** $nIK$ is sound and complete for $IK$. [Straßburger, 2013]

**Note:** A system can also be designed using labelled sequents. [Simpson, 1994]
Classical modal logic S5 is obtained from classical propositional logic

- by adding the *necessitation rule*: □A is a theorem if A is a theorem;
- and the axioms:

  k: □(A ⊃ B) ⊃ (□A ⊃ □B)
  t: A ⊃ ◯A
  4: ◯◇A ⊃ ◯A
  5: ◯A ⊃ ◯◇A
Sequent system:

\[
\begin{align*}
\Diamond_t^o & \frac{\Gamma, A}{\Gamma, \Diamond A} \\
\Box_{k45}^o & \frac{\Diamond \Gamma_1, \Gamma_1, \Box \Gamma_2, A}{\Diamond \Gamma_1, \Box \Gamma_2, \Gamma_3, \Box A}
\end{align*}
\]
Sequent system:

\[ \diamondsuit_t \frac{\Gamma, A}{\Gamma, \diamond A} \quad \Box^{\odot}_{k45} \frac{\diamondsuit_1, \Gamma, \Box \Gamma_2, A}{\diamondsuit_1, \Box \Gamma_2, \Gamma_3, \Box A} \]

Example:

\[
\diamondsuit_t \frac{\Diamond, a}{\Box, \Box a, \Diamond a} \quad \Box^{\odot}_{k45} \frac{\Box a, \Diamond a}{\Box a, \Box a, \Box \Diamond a}
\]

\[\Rightarrow\]

\[\Diamondsuit \frac{\Diamond a, \Box \Diamond a, \Box a}{\Box a} \quad \Box^{\odot}_{k45} \frac{\Box a}{\Box a, \Box a, \Box \Diamond a}
\]

Note: A cut-free system can also be achieved using hypersequents [Avron, 1996] or labelled sequents [Negri, 2005].
Classical modal logic S5

Sequent system:

\[
\begin{align*}
\Diamond_t \frac{\Gamma, A}{\Gamma, \Diamond A} & \quad \Box_{k45} \frac{\Diamond \Gamma_1, \Gamma_1, \Box \Gamma_2, A}{\Diamond \Gamma_1, \Box \Gamma_2, \Gamma_3, \Box A} \\
\end{align*}
\]

Nested sequent system: [Brünnler, 2009]

\[
\begin{align*}
\Box^n \frac{\Gamma \{[A]\}}{\Gamma \{\Box A\}} & \quad \Diamond^n \frac{\Gamma_1 \{\Diamond A, [A, \Gamma_2]\}}{\Gamma_1 \{\Diamond A, [\Gamma_2]\}} & \quad \Diamond^n \frac{\Gamma \{\Diamond A, A\}}{\Gamma \{\Diamond A\}} & \quad \Diamond^n \frac{\Gamma \{\Diamond A, [\Diamond A, \Gamma_2]\}}{\Gamma \{\Diamond A, [\Gamma_2]\}} & \quad \Diamond^n \frac{\Gamma_1 \{[\Diamond A, \Gamma_2]\}\{\Diamond A\}}{\Gamma_1 \{[\Diamond A, \Gamma_2]\}\{\emptyset\}} \\
\end{align*}
\]

Example:

\[
\begin{align*}
\Box^n \frac{\Diamond \overline{a}, \Box \Box a, a}{\overline{a}, \Box \Diamond a, \Diamond a} & \quad \Diamond^n \frac{\Box_{k45} \overline{a}, \Diamond a}{\overline{a}, \Box \Diamond a} & \quad \Diamond^n \frac{\Box_{k45} \Box \overline{a}, \Diamond a}{\overline{a}, \overline{a}, \Box \Diamond a} & \quad \Diamond^n \frac{\Box_{k45} \overline{a}, \Box \Box a, \Diamond a}{\overline{a}, \Box \Diamond a} \quad \text{cut}^o \\
\end{align*}
\]

Note: A cut-free system can also be achieved using hypersequents [Avron, 1996] or labelled sequents [Negri, 2005].
Classical modal logic S5

Sequent system:

\[
\begin{align*}
\Diamond_t \frac{}{\Gamma, \Diamond A} & \quad \Box_{k45}^o \frac{}{\Diamond \Gamma_1, \Box \Gamma_2, A} \\
\Gamma, \Diamond A & \quad \Box_{k45} \frac{}{\Diamond \Gamma_1, \Box \Gamma_2, \Gamma_3, \Box A}
\end{align*}
\]

Nested sequent system:

\[
\begin{align*}
\Box^n \frac{\Gamma \{ [A] \}}{\Gamma \{ \Box A \}} & \quad \Diamond^n \frac{\Gamma_1 \{ \Diamond A, [A, \Gamma_2] \}}{\Gamma_1 \{ \Diamond A, [\Gamma_2] \}} \\
\Gamma \{ \Box A \} & \quad \Diamond^n \frac{\Gamma \{ \Diamond A, A \}}{\Gamma \{ \Diamond A \}} \\
\Gamma_1 \{ \Diamond A, A \} & \quad \Diamond^n_4 \frac{\Gamma_1 \{ \Diamond A, [\Diamond A, \Gamma_2] \}}{\Gamma_1 \{ \Diamond A, [\Gamma_2] \}} \\
\Gamma \{ \Box A \} & \quad \Diamond^n_5 \frac{\Gamma_1 \{ [\Diamond A, \Gamma_2] \} \{ \Diamond A \}}{\Gamma_1 \{ [\Diamond A, \Gamma_2] \} \{ \emptyset \}}
\end{align*}
\]

Example:

\[
\begin{align*}
\ax^n \frac{}{\bar{a}, a, \lbrack \rbrack} & \quad \Box_{k45}^o \frac{}{\bar{a}, \Diamond a, a} \\
\Diamond_t \frac{}{\bar{a}, \Box \Diamond a, a} & \quad \Diamond_{k45}^o \frac{}{\Box \bar{a}, \bar{a}, \Diamond a} \\
\cut^o \frac{\Diamond_t}{\bar{a}, \Box \Diamond a} & \quad \leadsto \quad \ax^n \frac{}{\bar{a}, \Box \Diamond a}
\end{align*}
\]

Note: A cut-free system can also be achieved using hypersequents [Avron, 1996] or labelled sequents [Negri, 2005].
Classical modal logic S5

Sequent system:

\[
\begin{align*}
\Diamond_t \frac{\Gamma, A}{\Gamma, \Diamond A} & \quad \Box_{\text{k45}} \frac{\Diamond \Gamma_1, \Gamma_1, \Box \Gamma_2, A}{\Diamond \Gamma_1, \Box \Gamma_2, \Gamma_3, \Box A}
\end{align*}
\]

Nested sequent system:

\[
\begin{align*}
\Box_n \frac{\Gamma \{ [A] \}}{\Gamma \{ \Box A \}} & \quad \Diamond_k \frac{\Gamma_1 \{ \Diamond A, [A, \Gamma_2] \}}{\Gamma_1 \{ \Diamond A, [\Gamma_2] \}} \\
\Diamond_t \frac{\Gamma \{ \Diamond A, A \}}{\Gamma \{ \Diamond A \}} & \quad \Diamond_4 \frac{\Gamma_1 \{ \Diamond A, [\Diamond A, \Gamma_2] \}}{\Gamma_1 \{ \Diamond A, [\Gamma_2] \}} \\
\Diamond_5 \frac{\Gamma_1 \{ [\Diamond A, \Gamma_2] \} \{ [A] \}}{\Gamma_1 \{ [\Diamond A, \Gamma_2] \} \{ \emptyset \}}
\end{align*}
\]

Example:

\[
\begin{align*}
\text{ax}^o & \frac{\Diamond_t \Diamond \Diamond a, a}{\Diamond_t \Diamond a, a} \\
\Box_{\text{k45}} & \frac{\Diamond_t \Diamond a, a}{\Box a, \Diamond a} \\
\Diamond_t & \frac{\Diamond a, \Box \Diamond a, \Diamond a}{\Box a, \Diamond a, \Box \Diamond a} \\
\text{cut}^o & \frac{\Diamond_t \Diamond a, \Box \Diamond a}{\Box a, \Diamond a}
\end{align*}
\]

Note: A cut-free system can also be achieved using hypersequents [Avron, 1996]
or labelled sequents [Negri, 2005]
Examples: temporal logics → always, epistemic logics → common knowledge, program logics → iteration, modal μ-calculus → arbitrary fixed points:

\[ A ::= \ldots | \Box A | \Diamond A | \mu X.A | \nu X.A \]
Modal fixed point logics

**Examples:** temporal logics → *always*, epistemic logics → *common knowledge*, program logics → *iteration*, modal $\mu$-calculus → *arbitrary fixed points*:

\[
A ::= \ldots | \Box A | \Diamond A | \mu X.A | \nu X.A
\]

**Sequent system:**

\[
\begin{align*}
\Box^o_k & \quad \frac{\Gamma_1, A}{\Diamond \Gamma_1, \Box A, \Gamma_2} \\
\mu^o & \quad \frac{\Gamma, A(\mu X.A)}{\Gamma, \mu X.A} \\
\nu^o & \quad \frac{\{\Gamma, \nu^n X.A\}_{n \geq 0}}{\Gamma, \nu X.A}
\end{align*}
\]

Theorem: Sound and cut-free complete wrt. the modal $\mu$-calculus semantics.

▶ How to eliminate cuts between $\mu$ and $\nu$?

Alternative: Replace $\mu^o$ with rules $\Gamma, \mu X.A, \mu_i X.A$ for each $i \geq 0$.

$\Box^o X \supset \Diamond X$ is not derivable!
Modal fixed point logics

Examples: temporal logics $\rightarrow$ always, epistemic logics $\rightarrow$ common knowledge, program logics $\rightarrow$ iteration, modal $\mu$-calculus $\rightarrow$ arbitrary fixed points:

$$A ::= \ldots | \Box A | \Diamond A | \mu X.A | \nu X.A$$

Sequent system:

\[\begin{array}{c}
\Box_k \frac{\Gamma_1, A}{\Diamond \Gamma_1, \Box A, \Gamma_2}
\end{array}\]
\[\begin{array}{c}
\mu \frac{\Gamma, A(\mu X.A)}{\Gamma, \mu X.A}
\end{array}\]
\[\begin{array}{c}
\nu \frac{\{\Gamma, \nu^n X.A\}_{n \geq 0}}{\Gamma, \nu X.A}
\end{array}\]

Theorem: Sound and cut-free complete wrt. the modal $\mu$-calculus semantics.
Modal fixed point logics

**Examples:** temporal logics → *always*, epistemic logics → *common knowledge*, program logics → *iteration*, modal *μ*-calculus → *arbitrary fixed points:*

\[ A ::= \ldots \mid \Box A \mid \Diamond A \mid \mu X . A \mid \nu X . A \]

**Sequent system:**

\[
\begin{array}{cc}
\Box_k & \frac{\Gamma_1, A}{\Diamond \Gamma_1, \Box A, \Gamma_2} \\
\mu^\circ & \frac{\Gamma, A(X.A)}{\Gamma, \mu X . A} \\
\nu^\circ & \frac{\{\Gamma, \nu^n X . A\}_{n \geq 0}}{\Gamma, \nu X . A}
\end{array}
\]

**Theorem:** Sound and cut-free complete wrt. the modal *μ*-calculus semantics.

- How to eliminate cuts between *μ* and *ν*?
Modal fixed point logics

**Examples:** temporal logics $\rightarrow$ **always**, epistemic logics $\rightarrow$ **common knowledge**, program logics $\rightarrow$ **iteration**, modal $\mu$-calculus $\rightarrow$ arbitrary fixed points:

$$A ::= \ldots | \Box A | \Diamond A | \mu X. A | \nu X. A$$

**Sequent system:**

$$\begin{align*}
\Box^o_k & \quad \Gamma_1, A \quad \mu^o & \quad \Gamma, A(\mu X. A) \quad \nu^o & \quad \{\Gamma, \nu^n X. A\}_{n \geq 0} \\
\Diamond & \quad \Gamma_1, \Box A, \Gamma_2 & \quad \Gamma, \mu X. A & \quad \Gamma, \nu X. A
\end{align*}$$

**Theorem:** Sound and cut-free complete wrt. the modal $\mu$-calculus semantics.

- How to eliminate cuts between $\mu$ and $\nu$?

**Alternative:** Replace $\mu^o$ with rules $\mu^o_i$ for each $i \geq 0$. 

$$\begin{align*}
\mu^o_i & \quad \Gamma, \mu X. A, \mu^i X. A & \quad \Gamma, \mu X. A
\end{align*}$$
Modal fixed point logics

Examples: temporal logics → always, epistemic logics → common knowledge, program logics → iteration, modal $\mu$-calculus → arbitrary fixed points:

$$A ::= \ldots | \Box A | \Diamond A | \mu X.A | \nu X.A$$

Sequent system:

$$\begin{array}{c}
\Box^o_k \frac{\Gamma_1, A}{\Diamond \Gamma_1, \Box A, \Gamma_2} \\
\mu^o \frac{\Gamma, A(\mu X.A)}{\Gamma, \mu X.A} \\
\nu^o \frac{\{\Gamma, \nu^n X.A\}_{n \geq 0}}{\Gamma, \nu X.A}
\end{array}$$

Theorem: Sound and cut-free complete wrt. the modal $\mu$-calculus semantics.

- How to eliminate cuts between $\mu$ and $\nu$?

Alternative: Replace $\mu^o$ with rules $\mu_i^o$ for each $i \geq 0$.

$$\begin{array}{c}
\nu X.\Box X \supset \Box \nu X.\Box X \text{ is not derivable!}
\end{array}$$
Modal fixed point logics

Nested sequent system: [Brünnler and Studer, 2012]

\[
\begin{align*}
\square^n \frac{\Gamma \{ [A] \}}{\Gamma \{ \square A \}} & \quad \Diamond^n \frac{\Gamma_1 \{ \Diamond A, [A, \Gamma_2] \}}{\Gamma_1 \{ \Diamond A, [\Gamma_2] \}} & \quad \mu^n \frac{\Gamma \{ \mu X.A, \mu^i X.A \}}{\Gamma \{ \mu X.A \}} & \quad \nu^n \frac{\{ \nu^i X.A \}_{i \geq 0}}{\Gamma \{ \nu X.A \}}
\end{align*}
\]
**Modal fixed point logics**

**Nested sequent system:**

\[
\begin{align*}
\Box^n & \Gamma \{[A]\} \quad \Box^n & \Gamma_1 \{\Box A, [A, \Gamma_2]\} \\
\Box_k & \Gamma_1 \{\Box A, [\Gamma_2]\} \\
\mu_i^n & \Gamma \{\mu X . A, \mu^i X . A\} \quad \nu^n & \Gamma \{\nu X . A\}
\end{align*}
\]

**Example:**

\[
\begin{align*}
&\mu X . \Box X , [\mu^i X . \Box X , \nu^i X . \Box X] \\
\mu X . \Box X , [\nu X . \Box X] & \quad \mu X . \Box X, [\nu X . \Box X] \\
\mu X . \Box X, [\nu X . \Box X] & \quad \mu X . \Box X, [\nu X . \Box X] \\
\nu^n & \mu X . \Box X, [\nu X . \Box X] \\
\Box^n & \mu X . \Box X, [\nu X . \Box X]
\end{align*}
\]

Note:

Subsumes the systems for common knowledge [Brünnler and Studer, 2009] and PDL [Hill and Poggiolesi, 2010] but only complete for a fragment of modal \(\mu\).
Modal fixed point logics

Nested sequent system:  

\[ \square^n \frac{\Gamma \{ [A] \}}{\Gamma \{ \Box A \}} \quad \Diamond_k^n \frac{\Gamma_1 \{ \Diamond A, [A, \Gamma_2] \}}{\Gamma_1 \{ \Diamond A, [(\Gamma_2) \} \}} \quad \mu_i^n \frac{\Gamma \{ \mu X.A, \mu^i X.A \}}{\Gamma \{ \mu X.A \}} \quad \nu^n \frac{\{ \nu^i X.A \}_{i \geq 0}}{\Gamma \{ \nu X.A \}} \]

Example:

\[ \mu X. \Diamond X, [\mu^i X. \Diamond X, \nu^i X. \Box X] \]

\[ \Diamond_k^n \frac{\mu X. \Diamond X, \Diamond \mu^i X. \Diamond X, [\nu^i X. \Box X]}{\mu X. \Diamond X, \mu^{i+1} X. \Diamond X, [\nu^i X. \Box X]} \]

\[ \mu_i^n \frac{\mu X. \Diamond X, [\nu^i X. \Box X]}{\mu X. \Diamond X, [\nu^X. \Box X]} \]

\[ \nu^n \frac{\mu X. \Diamond X, [\nu^i X. \Box X]}{\mu X. \Diamond X, [\nu^X. \Box X]} \]

\[ \square^n \frac{\mu X. \Diamond X, [\nu X. \Box X]}{\mu X. \Diamond X, [\nu X. \Box X]} \]

Note: Subsumes the systems for common knowledge [Brünnler and Studer, 2009] and PDL [Hill and Poggiolesi, 2010] but only complete for a fragment of modal \( \mu \).
Where will they take you?
Conclusion

To give them better design and clean meta-theory
Understand the links between different formalisms
And applications.
Conclusion

Catch them all!
Conclusion

Catch them all!

- Classical and intuitionistic normal modal logics

- Fixed-point logics

- Grammar logics

- Provability logics

- Conditional logics

- etc...

To give them better design and clean meta-theory

Understand the links between different formalisms

And applications.
Conclusion

Catch them all!

- Classical and intuitionistic normal modal logics
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