Nested sequents for modal logics and beyond

Sonia Marin

IT-University of Copenhagen

July 7, 2018

This presentation was made possible by grant *NPRP 097-988-1-178*, from the Qatar National Research Fund (a member of the Qatar Foundation). The statements made herein are solely the responsibility of the author.

Survey talk on nested sequents

What are nested sequents?

What are nested sequents?

What can they achieve?

What are nested sequents?

What can they achieve?

1. for logics without a sequent system
   - intuitionistic modal logic IK

What are nested sequents?

What can they achieve?

1. for logics without a sequent system
   - intuitionistic modal logic IK
2. for sequent systems without a cut-free version
   - classical modal logic S5

What are nested sequents?

What can they achieve?

1. for logics without a sequent system
   - intuitionistic modal logic IK
2. for sequent systems without a cut-free version
   - classical modal logic S5
3. for cut-free systems without a syntactic cut-elimination procedure
   - modal fixed-point logic

What are nested sequents?

What can they achieve?

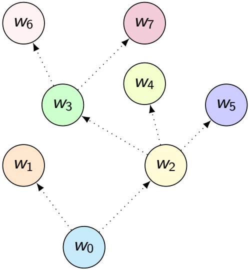
1. for logics without a sequent system
   - intuitionistic modal logic IK
2. for sequent systems without a cut-free version
   - classical modal logic S5
3. for cut-free systems without a syntactic cut-elimination procedure
   - modal fixed-point logic

Where will they take you?

What are nested sequents?

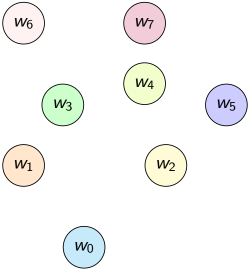
From semantics to syntax

Syntactical term encoding of semantical (tree) structure



[Brünnler, 2009] [Poggiolesi, 2009]

Syntactical term encoding of semantical (tree) structure

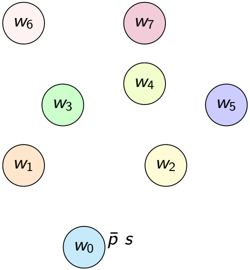


Nested sequents:

$\left[^{0} \quad \right]$

[Brünnler, 2009] [Poggiolesi, 2009]

Syntactical term encoding of semantical (tree) structure

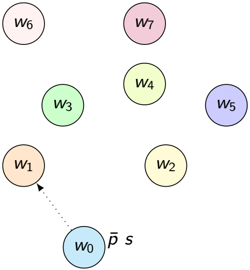


Nested sequents:

$$\left[{}^{0}\bar{p}, s, \ldots \right]$$

[Brünnler, 2009] [Poggiolesi, 2009]

Syntactical term encoding of semantical (tree) structure

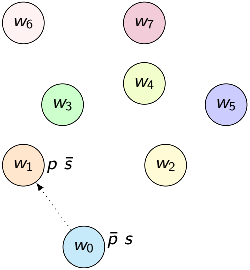


Nested sequents:

$$\left[^{0} \bar{p}, s, \ldots, \left[^{1} \quad \right] \quad \right]$$

[Brünnler, 2009] [Poggiolesi, 2009]

Syntactical term encoding of semantical (tree) structure

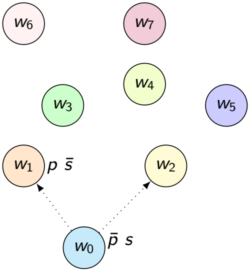


Nested sequents:

$$\left[^{0}\, \bar{p}, s, \ldots, \left[^{1}\, p, \bar{s}, \ldots\right] \qquad \right]$$

[Brünnler, 2009] [Poggiolesi, 2009]

Syntactical term encoding of semantical (tree) structure

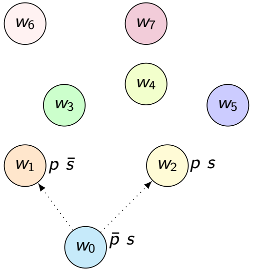


Nested sequents:

$$\left[{}^{0}\,\bar{p}, s, \ldots, \left[{}^{1}\,p, \bar{s}, \ldots\right], \left[{}^{2} \qquad\qquad \right]\right]$$

[Brünnler, 2009] [Poggiolesi, 2009]

Syntactical term encoding of semantical (tree) structure

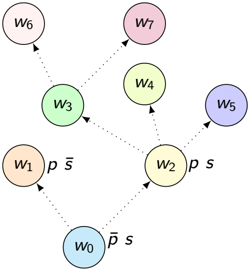


Nested sequents:

$$\left[^{0}\, \bar{p}, s, \ldots, \left[^{1}\, p, \bar{s}, \ldots\right], \left[^{2}\, p, s, \ldots \qquad \right]\right]$$

[Brünnler, 2009] [Poggiolesi, 2009]

Syntactical term encoding of semantical (tree) structure



Nested sequents:

$$\left[{}^{0}\bar{p}, s, \ldots, \left[{}^{1}p, \bar{s}, \ldots\right], \left[{}^{2}p, s, \ldots, \left[{}^{3}\ldots, \left[{}^{6}\ldots\right], \left[{}^{7}\ldots\right]\right], \left[{}^{4}\ldots\right], \left[{}^{5}\ldots\right]\right]\right]$$

[Brünnler, 2009] [Poggiolesi, 2009]

What can they achieve?

Intuitionistic modal logic IK

Intuitionistic modal logic IK is obtained from intuitionistic propositional logic

- by adding the *necessitation rule*: $\Box A$ is a theorem if A is a theorem;
- and the following five variants of the k axiom.

$$\begin{array}{llll} \mathsf{k}_1: & \Box(A \supset B) \supset (\Box A \supset \Box B) & \mathsf{k}_3: & \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B) \\ \mathsf{k}_2: & \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) & \mathsf{k}_4: & (\Diamond A \supset \Box B) \supset \Box(A \supset B) \\ & & \mathsf{k}_5: & \Diamond \bot \supset \bot \end{array}$$

Intuitionistic modal logic IK

Intuitionistic modal logic IK is obtained from intuitionistic propositional logic

- by adding the *necessitation rule*: $\Box A$ is a theorem if A is a theorem;
- and the following five variants of the k axiom.

$$\begin{array}{ll} \mathsf{k}_1: \Box(A \supset B) \supset (\Box A \supset \Box B) & \mathsf{k}_3: \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B) \\ \mathsf{k}_2: \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) & \mathsf{k}_4: (\Diamond A \supset \Box B) \supset \Box(A \supset B) \\ & \mathsf{k}_5: \Diamond\bot \supset \bot \end{array}$$

Sequent system:

$$\Box^{\circ}_{\mathsf{k}} \frac{\Lambda \Rightarrow A}{\Box\Lambda \Rightarrow \Box A} \qquad \Diamond^{\circ}_{\mathsf{k}} \frac{\Lambda, A \Rightarrow B}{\Box\Lambda, \Diamond A \Rightarrow \Diamond B}$$

Intuitionistic modal logic IK is obtained from intuitionistic propositional logic

- by adding the *necessitation rule*: $\Box A$ is a theorem if A is a theorem;
- and the following five variants of the k axiom.

$$\begin{array}{llll} \mathsf{k}_1: & \Box(A \supset B) \supset (\Box A \supset \Box B) & \mathsf{k}_3: & \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B) \\ \mathsf{k}_2: & \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) & \mathsf{k}_4: & (\Diamond A \supset \Box B) \supset \Box(A \supset B) \\ & & \mathsf{k}_5: & \Diamond\bot \supset \bot \end{array}$$

Sequent system:

$$\Box^\circ_k \frac{\Lambda \Rightarrow A}{\Box\Lambda \Rightarrow \Box A} \qquad \Diamond^\circ_k \frac{\Lambda, A \Rightarrow B}{\Box\Lambda, \Diamond A \Rightarrow \Diamond B}$$

?

Theorem: $\mathsf{LJ_p} + \Box^\circ_k + \Diamond^\circ_k$ is sound and complete for $\mathsf{IK} - \{\mathsf{k}_3, \mathsf{k}_4, \mathsf{k}_5\}$.

Intuitionistic modal logic IK is obtained from intuitionistic propositional logic

- by adding the *necessitation rule*: $\Box A$ is a theorem if A is a theorem;
- and the following five variants of the k axiom.

$$\begin{array}{llll} \mathsf{k}_1: & \Box(A \supset B) \supset (\Box A \supset \Box B) & \mathsf{k}_3: & \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B) \\ \mathsf{k}_2: & \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) & \mathsf{k}_4: & (\Diamond A \supset \Box B) \supset \Box(A \supset B) \\ & & \mathsf{k}_5: & \Diamond\bot \supset \bot \end{array}$$

Nested sequent system:

$$\Diamond^{\mathsf{n}}_{\mathsf{Rk}}\ \frac{\Lambda_1\{[\Lambda_2, A]\}}{\Lambda_1\{[\Lambda_2], \Diamond A\}} \qquad \Diamond^{\mathsf{n}}_{\mathsf{L}}\ \frac{\Pi\{[A]\}}{\Pi\{\Diamond A\}} \qquad \Box^{\mathsf{n}}_{\mathsf{R}}\ \frac{\Lambda\{[A]\}}{\Lambda\{\Box A\}} \qquad \Box^{\mathsf{n}}_{\mathsf{Lk}}\ \frac{\Delta_1\{\Box A, [A, \Delta_2]\}}{\Delta_1\{\Box A, [\Delta_2]\}}$$

Intuitionistic modal logic IK is obtained from intuitionistic propositional logic

- by adding the *necessitation rule*: $\Box A$ is a theorem if A is a theorem;
- and the following five variants of the k axiom.

$$\mathsf{k}_1: \ \Box(A \supset B) \supset (\Box A \supset \Box B) \qquad \mathsf{k}_3: \ \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)$$
$$\mathsf{k}_2: \ \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) \qquad \mathsf{k}_4: \ (\Diamond A \supset \Box B) \supset \Box(A \supset B)$$
$$\mathsf{k}_5: \ \Diamond \bot \supset \bot$$

Nested sequent system:

$$\Diamond^{\mathsf{n}}_{\mathsf{Rk}} \frac{\Lambda_1\{[\Lambda_2, A]\}}{\Lambda_1\{[\Lambda_2], \Diamond A\}} \quad \Diamond^{\mathsf{n}}_{\mathsf{L}} \frac{\Pi\{[A]\}}{\Pi\{\Diamond A\}} \quad \Box^{\mathsf{n}}_{\mathsf{R}} \frac{\Lambda\{[A]\}}{\Lambda\{\Box A\}} \quad \Box^{\mathsf{n}}_{\mathsf{Lk}} \frac{\Delta_1\{\Box A, [A, \Delta_2]\}}{\Delta_1\{\Box A, [\Delta_2]\}}$$

Theorem: nIK is sound and complete for IK. [Straßburger, 2013]

Intuitionistic modal logic IK is obtained from intuitionistic propositional logic

- by adding the *necessitation rule*: $\Box A$ is a theorem if A is a theorem;
- and the following five variants of the k axiom.

$$\mathsf{k}_1 : \ \Box(A \supset B) \supset (\Box A \supset \Box B)$$
$$\mathsf{k}_2 : \ \Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$$
$$\mathsf{k}_3 : \ \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)$$
$$\mathsf{k}_4 : \ (\Diamond A \supset \Box B) \supset \Box(A \supset B)$$
$$\mathsf{k}_5 : \ \Diamond\bot \supset \bot$$

Nested sequent system:

$$\Diamond^{\mathsf{n}}_{\mathsf{Rk}} \frac{\Lambda_1\{[\Lambda_2, A]\}}{\Lambda_1\{[\Lambda_2], \Diamond A\}} \quad \Diamond^{\mathsf{n}}_{\mathsf{L}} \frac{\Pi\{[A]\}}{\Pi\{\Diamond A\}} \quad \Box^{\mathsf{n}}_{\mathsf{R}} \frac{\Lambda\{[A]\}}{\Lambda\{\Box A\}} \quad \Box^{\mathsf{n}}_{\mathsf{Lk}} \frac{\Delta_1\{\Box A, [A, \Delta_2]\}}{\Delta_1\{\Box A, [\Delta_2]\}}$$

Theorem: nIK is sound and complete for IK. [Straßburger, 2013]

Note: A system can also be designed using labelled sequents. [Simpson, 1994]

Classical modal logic S5

Classical modal logic S5 is obtained from classical propositional logic

- by adding the *necessitation rule*: $\Box A$ is a theorem if A is a theorem;
- and the axioms:

$$
\begin{aligned}
\mathsf{k}: &\quad \Box(A \supset B) \supset (\Box A \supset \Box B) \\
\mathsf{t}: &\quad A \supset \Diamond A \\
4: &\quad \Diamond\Diamond A \supset \Diamond A \\
5: &\quad \Diamond A \supset \Box\Diamond A
\end{aligned}
$$

Sequent system:

$$\Diamond^{\circ}_{\mathsf{t}}\ \frac{\Gamma, A}{\Gamma, \Diamond A} \qquad \Box^{\circ}_{\mathsf{k45}}\ \frac{\Diamond\Gamma_1, \Gamma_1, \Box\Gamma_2, A}{\Diamond\Gamma_1, \Box\Gamma_2, \Gamma_3, \Box A}$$

Sequent system:

$$\Diamond^{\circ}_{t}\ \frac{\Gamma, A}{\Gamma, \Diamond A} \qquad \Box^{\circ}_{k45}\ \frac{\Diamond\Gamma_1, \Gamma_1, \Box\Gamma_2, A}{\Diamond\Gamma_1, \Box\Gamma_2, \Gamma_3, \Box A}$$

Example:

$$= \ {\color{red}\mathsf{cut}^{\circ}}\ \frac{\Diamond^{\circ}_{t}\ \dfrac{\mathsf{ax}^{\circ}\ \dfrac{}{\bar{a}, \Box\Diamond a, a}}{\bar{a}, \Box\Diamond a, \Diamond a} \qquad \Box^{\circ}_{k45}\ \dfrac{\Box^{\circ}_{k45}\ \dfrac{\mathsf{ax}^{\circ}\ \dfrac{}{\bar{a}, a}}{\Box\bar{a}, \Diamond a}}{\Box\bar{a}, \bar{a}, \Box\Diamond a}}{\bar{a}, \Box\Diamond a}$$

Classical modal logic S5

Sequent system:

$$\Diamond^{\mathsf{o}}_{\mathsf{t}}\ \frac{\Gamma, A}{\Gamma, \Diamond A} \qquad \Box^{\mathsf{o}}_{\mathsf{k45}}\ \frac{\Diamond\Gamma_1, \Gamma_1, \Box\Gamma_2, A}{\Diamond\Gamma_1, \Box\Gamma_2, \Gamma_3, \Box A}$$

Nested sequent system: [Brünnler, 2009]

$$\Box^{\mathsf{n}}\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \quad \Diamond^{\mathsf{n}}_{\mathsf{k}}\ \frac{\Gamma_1\{\Diamond A, [A, \Gamma_2]\}}{\Gamma_1\{\Diamond A, [\Gamma_2]\}} \quad \Diamond^{\mathsf{n}}_{\mathsf{t}}\ \frac{\Gamma\{\Diamond A, A\}}{\Gamma\{\Diamond A\}} \quad \Diamond^{\mathsf{n}}_{4}\ \frac{\Gamma_1\{\Diamond A, [\Diamond A, \Gamma_2]\}}{\Gamma_1\{\Diamond A, [\Gamma_2]\}} \quad \Diamond^{\mathsf{n}}_{5}\ \frac{\Gamma_1\{[\Diamond A, \Gamma_2]\}\{\Diamond A\}}{\Gamma_1\{[\Diamond A, \Gamma_2]\}\{\emptyset\}}$$

Example:

$$= \ \mathsf{cut}^{\mathsf{o}}\ \frac{\Diamond^{\mathsf{o}}_{\mathsf{t}}\ \dfrac{\mathsf{ax}^{\mathsf{o}}\ \dfrac{}{\bar{a}, \Box\Diamond a, a}}{\bar{a}, \Box\Diamond a, \Diamond a} \qquad \Box^{\mathsf{o}}_{\mathsf{k45}}\ \dfrac{\Box^{\mathsf{o}}_{\mathsf{k45}}\ \dfrac{\mathsf{ax}^{\mathsf{o}}\ \dfrac{}{\bar{a}, a}}{\Box\bar{a}, \Diamond a}}{\Box\bar{a}, \bar{a}, \Box\Diamond a}}{\bar{a}, \Box\Diamond a}$$

Sequent system:

$$\Diamond^{\mathsf{o}}_{\mathsf{t}}\ \frac{\Gamma, A}{\Gamma, \Diamond A} \qquad \Box^{\mathsf{o}}_{\mathsf{k45}}\ \frac{\Diamond\Gamma_1, \Gamma_1, \Box\Gamma_2, A}{\Diamond\Gamma_1, \Box\Gamma_2, \Gamma_3, \Box A}$$

Nested sequent system: [Brünnler, 2009]

$$\Box^{\mathsf{n}}\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \quad \Diamond^{\mathsf{n}}_{\mathsf{k}}\ \frac{\Gamma_1\{\Diamond A, [A, \Gamma_2]\}}{\Gamma_1\{\Diamond A, [\Gamma_2]\}} \quad \Diamond^{\mathsf{n}}_{\mathsf{t}}\ \frac{\Gamma\{\Diamond A, A\}}{\Gamma\{\Diamond A\}} \quad \Diamond^{\mathsf{n}}_{4}\ \frac{\Gamma_1\{\Diamond A, [\Diamond A, \Gamma_2]\}}{\Gamma_1\{\Diamond A, [\Gamma_2]\}} \quad \Diamond^{\mathsf{n}}_{5}\ \frac{\Gamma_1\{[\Diamond A, \Gamma_2]\}\{\Diamond A\}}{\Gamma_1\{[\Diamond A, \Gamma_2]\}\{\varnothing\}}$$

Example:

$$= \mathsf{cut}^{\mathsf{o}}\ \dfrac{\Diamond^{\mathsf{o}}_{\mathsf{t}}\ \dfrac{\mathsf{ax}^{\mathsf{o}}\ \dfrac{}{\bar{a}, \Box\Diamond a, a}}{\bar{a}, \Box\Diamond a, \Diamond a} \qquad \Box^{\mathsf{o}}_{\mathsf{k45}}\ \dfrac{\Box^{\mathsf{o}}_{\mathsf{k45}}\ \dfrac{\mathsf{ax}^{\mathsf{o}}\ \dfrac{}{\bar{a}, a}}{\Box\bar{a}, \Diamond a}}{\Box\bar{a}, \bar{a}, \Box\Diamond a}}{\bar{a}, \Box\Diamond a} \quad \rightsquigarrow \quad = \Box^{\mathsf{n}}\ \dfrac{\Diamond^{\mathsf{n}}_{5}\ \dfrac{\Diamond^{\mathsf{n}}_{\mathsf{t}}\ \dfrac{\mathsf{ax}^{\mathsf{n}}\ \dfrac{}{\bar{a}, a, [\,]}}{\bar{a}, \Diamond a, [\,]}}{\bar{a}, [\Diamond a]}}{\bar{a}, \Box\Diamond a}$$

Sequent system:

$$\Diamond^{\mathsf{o}}_{\mathsf{t}}\ \frac{\Gamma, A}{\Gamma, \Diamond A} \qquad \Box^{\mathsf{o}}_{\mathsf{k45}}\ \frac{\Diamond\Gamma_1, \Gamma_1, \Box\Gamma_2, A}{\Diamond\Gamma_1, \Box\Gamma_2, \Gamma_3, \Box A}$$

Nested sequent system: [Brünnler, 2009]

$$\Box^{\mathsf{n}}\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \quad \Diamond^{\mathsf{n}}_{\mathsf{k}}\ \frac{\Gamma_1\{\Diamond A, [A, \Gamma_2]\}}{\Gamma_1\{\Diamond A, [\Gamma_2]\}} \quad \Diamond^{\mathsf{n}}_{\mathsf{t}}\ \frac{\Gamma\{\Diamond A, A\}}{\Gamma\{\Diamond A\}} \quad \Diamond^{\mathsf{n}}_{4}\ \frac{\Gamma_1\{\Diamond A, [\Diamond A, \Gamma_2]\}}{\Gamma_1\{\Diamond A, [\Gamma_2]\}} \quad \Diamond^{\mathsf{n}}_{5}\ \frac{\Gamma_1\{[\Diamond A, \Gamma_2]\}\{\Diamond A\}}{\Gamma_1\{[\Diamond A, \Gamma_2]\}\{\varnothing\}}$$

Example:

$$= \ \mathsf{cut}^{\mathsf{o}}\ \dfrac{\Diamond^{\mathsf{o}}_{\mathsf{t}}\ \dfrac{\mathsf{ax}^{\mathsf{o}}\ \dfrac{}{\bar{a}, \Box\Diamond a, a}}{\bar{a}, \Box\Diamond a, \Diamond a} \qquad \Box^{\mathsf{o}}_{\mathsf{k45}}\ \dfrac{\Box^{\mathsf{o}}_{\mathsf{k45}}\ \dfrac{\mathsf{ax}^{\mathsf{o}}\ \dfrac{}{\bar{a}, a}}{\Box\bar{a}, \Diamond a}}{\Box\bar{a}, \bar{a}, \Box\Diamond a}}{\bar{a}, \Box\Diamond a} \quad \rightsquigarrow \quad = \ \Box^{\mathsf{n}}\ \dfrac{\Diamond^{\mathsf{n}}_{5}\ \dfrac{\Diamond^{\mathsf{n}}_{\mathsf{t}}\ \dfrac{\mathsf{ax}^{\mathsf{n}}\ \dfrac{}{\bar{a}, a, [\,]}}{\bar{a}, \Diamond a, [\,]}}{\bar{a}, [\Diamond a]}}{\bar{a}, \Box\Diamond a}$$

Note: A cut-free system can also be achieved using hypersequents [Avron, 1996] or labelled sequents [Negri, 2005]

Modal fixed point logics

Examples: temporal logics $\to$ always, epistemic logics $\to$ common knowledge, program logics $\to$ iteration, modal μ-calculus $\to$ arbitrary fixed points:

$$A ::= \ldots \mid \Box A \mid \Diamond A \mid \mu X.A \mid \nu X.A$$

Modal fixed point logics

Examples: temporal logics $\to$ always, epistemic logics $\to$ common knowledge, program logics $\to$ iteration, modal μ-calculus $\to$ arbitrary fixed points:

$$A ::= \ldots \mid \Box A \mid \Diamond A \mid \mu X.A \mid \nu X.A$$

Sequent system:

$$\Box_k^\circ \, \frac{\Gamma_1, A}{\Diamond\Gamma_1, \Box A, \Gamma_2} \quad \mu^\circ \, \frac{\Gamma, A(\mu X.A)}{\Gamma, \mu X.A} \quad \nu^\circ \, \frac{\{\Gamma, \nu^n X.A\}_{n\geq 0}}{\Gamma, \nu X.A}$$

Examples: temporal logics $\to$ always, epistemic logics $\to$ common knowledge, program logics $\to$ iteration, modal μ-calculus $\to$ arbitrary fixed points:

$$A ::= \ldots \mid \Box A \mid \Diamond A \mid \mu X.A \mid \nu X.A$$

Sequent system:

$$\Box_k^\circ \frac{\Gamma_1, A}{\Diamond\Gamma_1, \Box A, \Gamma_2} \quad \mu^\circ \frac{\Gamma, A(\mu X.A)}{\Gamma, \mu X.A} \quad \nu^\circ \frac{\{\Gamma, \nu^n X.A\}_{n\geq 0}}{\Gamma, \nu X.A}$$

Theorem: Sound and cut-free complete wrt. the modal μ-calculus semantics.

Modal fixed point logics

Examples: temporal logics → always, epistemic logics → common knowledge, program logics → iteration, modal μ-calculus → arbitrary fixed points:

$$A ::= \ldots \mid \Box A \mid \Diamond A \mid \mu X.A \mid \nu X.A$$

Sequent system:

$$\Box_k^\circ \frac{\Gamma_1, A}{\Diamond\Gamma_1, \Box A, \Gamma_2} \quad \mu^\circ \frac{\Gamma, A(\mu X.A)}{\Gamma, \mu X.A} \quad \nu^\circ \frac{\{\Gamma, \nu^n X.A\}_{n\geq 0}}{\Gamma, \nu X.A}$$

Theorem: Sound and cut-free complete wrt. the modal μ-calculus semantics.

- How to eliminate cuts between μ and ν?

Modal fixed point logics

Examples: temporal logics → always, epistemic logics → common knowledge, program logics → iteration, modal μ-calculus → arbitrary fixed points:

$$A ::= \ldots \mid \Box A \mid \Diamond A \mid \mu X.A \mid \nu X.A$$

Sequent system:

$$\Box^{\circ}_{\mathsf{k}} \ \frac{\Gamma_1, A}{\Diamond\Gamma_1, \Box A, \Gamma_2} \qquad \mu^{\circ} \ \frac{\Gamma, A(\mu X.A)}{\Gamma, \mu X.A} \qquad \nu^{\circ} \ \frac{\{\Gamma, \nu^n X.A\}_{n \geq 0}}{\Gamma, \nu X.A}$$

Theorem: Sound and cut-free complete wrt. the modal μ-calculus semantics.

- How to eliminate cuts between μ and ν?

Alternative: Replace μ° with rules $\mu^{\circ}_{\mathsf{i}} \ \frac{\Gamma, \mu X.A, \mu^i X.A}{\Gamma, \mu X.A}$ for each $i \geq 0$.

Modal fixed point logics

Examples: temporal logics → always, epistemic logics → common knowledge, program logics → iteration, modal μ-calculus → arbitrary fixed points:

$$A ::= \ldots \mid \Box A \mid \Diamond A \mid \mu X.A \mid \nu X.A$$

Sequent system:

$$\Box^{\circ}_{\mathsf{k}} \; \frac{\Gamma_1, A}{\Diamond\Gamma_1, \Box A, \Gamma_2} \qquad \mu^{\circ} \; \frac{\Gamma, A(\mu X.A)}{\Gamma, \mu X.A} \qquad \nu^{\circ} \; \frac{\{\Gamma, \nu^n X.A\}_{n\geq 0}}{\Gamma, \nu X.A}$$

Theorem: Sound and cut-free complete wrt. the modal μ-calculus semantics.

- How to eliminate cuts between μ and ν?

Alternative: Replace μ° with rules $\mu^{\circ}_{\mathsf{i}} \; \dfrac{\Gamma, \mu X.A, \mu^i X.A}{\Gamma, \mu X.A}$ for each $i \geq 0$.

- $\nu X.\Box X \supset \Box\nu X.\Box X$ is not derivable!

Modal fixed point logics

Nested sequent system: [Brünnler and Studer, 2012]

$$\Box^{\mathsf{n}} \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \quad \Diamond_{\mathsf{k}}^{\mathsf{n}} \frac{\Gamma_1\{\Diamond A, [A, \Gamma_2]\}}{\Gamma_1\{\Diamond A, [\Gamma_2]\}} \quad \mu_{\mathsf{i}}^{\mathsf{n}} \frac{\Gamma\{\mu X.A, \mu^i X.A\}}{\Gamma\{\mu X.A\}} \quad \nu^{\mathsf{n}} \frac{\{\nu^i X.A\}_{i \geq 0}}{\Gamma\{\nu X.A\}}$$

Nested sequent system: [Brünnler and Studer, 2012]

$$\square^{\mathsf{n}}\ \frac{\Gamma\{[A]\}}{\Gamma\{\square A\}} \qquad \Diamond_{\mathsf{k}}^{\mathsf{n}}\ \frac{\Gamma_1\{\Diamond A, [A, \Gamma_2]\}}{\Gamma_1\{\Diamond A, [\Gamma_2]\}} \qquad \mu_{\mathsf{i}}^{\mathsf{n}}\ \frac{\Gamma\{\mu X.A, \mu^i X.A\}}{\Gamma\{\mu X.A\}} \qquad \nu^{\mathsf{n}}\ \frac{\{\nu^i X.A\}_{i\geq 0}}{\Gamma\{\nu X.A\}}$$

Example:

$$\nu^{\mathsf{n}}\ \frac{\mu X.\Diamond X, \left[\mu^i X.\Diamond X, \nu^i X.\square X\right] \quad \cdots \quad \mu_{\mathsf{i+1}}^{\mathsf{n}}\ \dfrac{=\ \dfrac{\Diamond_{\mathsf{k}}^{\mathsf{n}}\ \dfrac{}{\mu X.\Diamond X, \Diamond\mu^i X.\Diamond X, \left[\nu^i X.\square X\right]}}{\mu X.\Diamond X, \mu^{i+1} X.\Diamond X, \left[\nu^i X.\square X\right]}}{\mu X.\Diamond X, \left[\nu^i X.\square X\right]} \quad \cdots}{\square^{\mathsf{n}}\ \dfrac{\mu X.\Diamond X, [\nu X.\square X]}{\mu X.\Diamond X, \square\nu X.\square X}}$$

Nested sequent system: [Brünnler and Studer, 2012]

$$\Box^{\mathsf{n}}\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \Diamond^{\mathsf{n}}_{\mathsf{k}}\ \frac{\Gamma_1\{\Diamond A, [A, \Gamma_2]\}}{\Gamma_1\{\Diamond A, [\Gamma_2]\}} \qquad \mu^{\mathsf{n}}_{\mathsf{i}}\ \frac{\Gamma\{\mu X.A, \mu^i X.A\}}{\Gamma\{\mu X.A\}} \qquad \nu^{\mathsf{n}}\ \frac{\{\nu^i X.A\}_{i\geq 0}}{\Gamma\{\nu X.A\}}$$

Example:

$$\Box^{\mathsf{n}}\ \cfrac{\nu^{\mathsf{n}}\ \cfrac{\mu X.\Diamond X, \left[\mu^i X.\Diamond X, \nu^i X.\Box X\right] \quad \cdots \quad \mu^{\mathsf{n}}_{\mathsf{i+1}}\ \cfrac{= \cfrac{\Diamond^{\mathsf{n}}_{\mathsf{k}}\ \cfrac{}{\mu X.\Diamond X, \Diamond\mu^i X.\Diamond X, \left[\nu^i X.\Box X\right]}}{\mu X.\Diamond X, \mu^{i+1} X.\Diamond X, \left[\nu^i X.\Box X\right]}}{\mu X.\Diamond X, \left[\nu^i X.\Box X\right]} \quad \cdots}{\mu X.\Diamond X, [\nu X.\Box X]}}{\mu X.\Diamond X, \Box\nu X.\Box X}$$

Note: Subsumes the systems for common knowledge [Brünnler and Studer, 2009] and PDL [Hill and Poggiolesi, 2010] but only complete for a fragment of modal μ.

Where will they take you?

Conclusion

Conclusion

Catch them all!

Conclusion

Catch them all!

- Classical and intuitionistic normal modal logics

Conclusion

Catch them all!

- Classical and intuitionistic normal modal logics
- Fixed-point logics

Conclusion

Catch them all!

- Classical and intuitionistic normal modal logics
- Fixed-point logics
- Grammar logics [Tiu, Ianovski, and Goré, 2012]

Conclusion

Catch them all!

- Classical and intuitionistic normal modal logics
- Fixed-point logics
- Grammar logics [Tiu, Ianovski, and Goré, 2012]
- Provability logics [Poggiolesi, 2009], [Shamkanov, 2015]

Conclusion

Catch them all!

- Classical and intuitionistic normal modal logics
- Fixed-point logics
- Grammar logics [Tiu, Ianovski, and Goré, 2012]
- Provability logics [Poggiolesi, 2009], [Shamkanov, 2015]
- Conditional logics [Alenda, Olivetti, and Pozzato, 2012]

Conclusion

Catch them all!

- Classical and intuitionistic normal modal logics
- Fixed-point logics
- Grammar logics [Tiu, Ianovski, and Goré, 2012]
- Provability logics [Poggiolesi, 2009], [Shamkanov, 2015]
- Conditional logics [Alenda, Olivetti, and Pozzato, 2012]
- etc...

Conclusion

Catch them all!

- Classical and intuitionistic normal modal logics
- Fixed-point logics
- Grammar logics [Tiu, Ianovski, and Goré, 2012]
- Provability logics [Poggiolesi, 2009], [Shamkanov, 2015]
- Conditional logics [Alenda, Olivetti, and Pozzato, 2012]
- etc...

To give them better design and clean meta-theory

Conclusion

Catch them all!

- Classical and intuitionistic normal modal logics
- Fixed-point logics
- Grammar logics [Tiu, Ianovski, and Goré, 2012]
- Provability logics [Poggiolesi, 2009], [Shamkanov, 2015]
- Conditional logics [Alenda, Olivetti, and Pozzato, 2012]
- etc...

To give them better design and clean meta-theory

Understand the links between different formalisms

Conclusion

Catch them all!

- Classical and intuitionistic normal modal logics
- Fixed-point logics
- Grammar logics [Tiu, Ianovski, and Goré, 2012]
- Provability logics [Poggiolesi, 2009], [Shamkanov, 2015]
- Conditional logics [Alenda, Olivetti, and Pozzato, 2012]
- etc...

To give them better design and clean meta-theory

Understand the links between different formalisms

And applications.