Comparing $\Box$ and ! via polarities

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May 19, 2017

The answer

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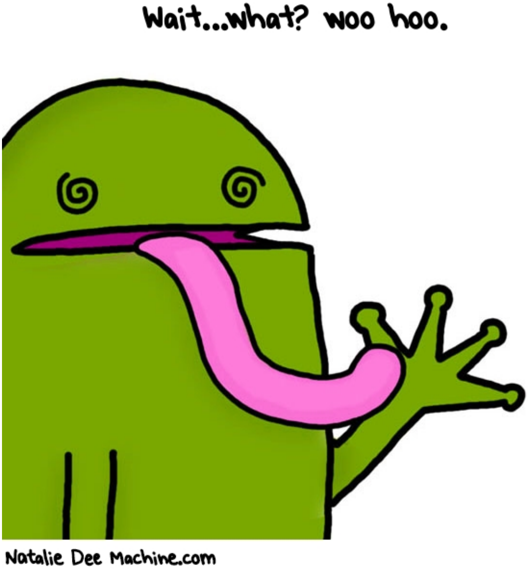
"from a proof-theoretical point of view
exponentials behave exactly like S4 modalities"

[Martini & Masini, 1994]


Wait...what? woo hoo.
Natalie Dee Machine.com

Are ! and □ interchangeable?

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Modal logic S4**:**

$A ::= x \; j \; x^{\perp} \; j \; A \;\hat{}\; A \; j \; > \; j \; A \; \_ \; A \; j \; ?$

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$A ::= x \mid x^{\perp} \mid A \wedge A \mid \top \mid A \vee A \mid \bot \mid \Box A \mid \Diamond A$

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Linear logic LL**:**
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$$\Box\ \frac{\vdash \Diamond\Gamma, A}{\vdash \Diamond\Gamma, \Box A, \Delta} \qquad \Diamond\ \frac{\vdash \Gamma, \Diamond A, A}{\vdash \Gamma, \Diamond A}$$

Linear logic LL**:**

$A ::= x \mid x^\perp \mid A \otimes A \mid 1 \mid A \parr A \mid \bot \mid A \oplus A \mid 0 \mid A \,\&\, A \mid \top \mid\ !A \mid ?A$

$$!\ \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A} \qquad ?\ \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}$$

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Are ! and □ interchangeable?

Theorem: [Martini & Masini, 1994]

Γ provable in S4 , Γ^+ provable in LL

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$$\Box \; \frac{` \; \Diamond\Gamma, A}{` \; \Diamond\Gamma, \Box A, \Delta} \qquad \Diamond \; \frac{` \; \Gamma, \Diamond A, A}{` \; \Gamma, \Diamond A}$$

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$A ::= x \; j \; x^\perp \; j \; A \quad A \; j \; 1 \; j \; A \quad A \; j \; ? \; j \; A \quad A \; j \; 0 \; j \; A \,\&\, A \; j \; > \; j \; !A \; j \; ?A$

$$! \; \frac{` \; ?\Gamma, A}{` \; ?\Gamma, !A} \qquad ? \; \frac{` \; \Gamma, A}{` \; \Gamma, ?A}$$

Are ! and $\Box$ interchangeable?

Theorem: [Martini & Masini, 1994]

$$\Gamma \text{ provable in S4} \quad , \quad \Gamma^{+} \text{ provable in LL}$$

Are ! and □ interchangeable?

Theorem: [Martini & Masini, 1994]

Γ provable in S4 , Γ^+ provable in LL

Their answer:

cut-free proof of an $S4$ sequent

m

cut-free proof of its LL translation

Are ! and □ interchangeable?

Theorem: [Martini & Masini, 1994]

Γ provable in S4 , Γ^+ provable in LL

Our question:

focused polarised
cut-free proof of an $S4$ sequent
m
focused polarised
cut-free proof of its LL translation

?

Polarity and focusing

Polarities:	non-invertible rules	:	positive connectives
	invertible rules	:	negative connectives

[Andreoli, 1990] [Laurent, 2004]

Polarity and focusing

Polarities: non-invertible rules : positive connectives
invertible rules : negative connectives

Inversion: in $\dfrac{\pi \Big\|}{\vdash N, \Gamma}$ the last rule is negative.

[Andreoli, 1990] [Laurent, 2004]

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Focus on a positive formula:

in $\dfrac{\pi \Vert}{\vdash P, \Gamma}$ only rules decomposing P between two rules decomposing P

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Completeness of focusing:

if a formula F is provable then F has a focused proof

[Andreoli, 1990] [Laurent, 2004]

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$A ::= x \ j \ x^{\perp} \ j \ A \quad A \ j \ 1 \ j \ A \quad A \ j \ ? \ j \ A \quad A \ j \ 0 \ j \ A \,\&\, A \ j > j \ !A \ j \ ?A$

$P \quad ::= \quad x \quad j \ A \quad A \ j \ 1 \ j \ A \quad A \ j \ 0$
$N \quad ::= \quad x^{\perp} \ j \ A \quad A \ j \ ? \ j \ A \,\&\, A \ j >$

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$$P \quad ::= \quad x \;\; j \; A \overset{+}{\wedge} A \; j \; \overset{+}{\top} \; j \; A \overset{+}{\vee} A \; j \; \overset{+}{\bot}$$

$$N \quad ::= \quad x^{\perp} j \; A \overset{-}{\vee} A \; j \; \overset{-}{\bot} \; j \; A \overset{-}{\wedge} A \; j \; \overset{-}{\top}$$

Linear logic LL**:**

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$P \quad ::= \quad x \;\; j \; A \overset{+}{\wedge} A \; j \; \overset{+}{\top} \; j \; A \overset{+}{\vee} A \; j \; \overset{+}{\bot} \; j \; \Diamond \, A$

$N \quad ::= \quad x^{\perp} \, j \; A \overset{-}{\vee} A \; j \; \overset{-}{\bot} \; j \; A \overset{-}{\wedge} A \; j \; \overset{-}{\top} \; j \; \Box \, A$

[Miller, Volpe, 2015] [Chaudhuri, M., Strassburger, 2016]

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$P \; ::= \quad j \; \Diamond \; A$

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This is...

[Miller, Volpe, 2015] [Chaudhuri, M., Strassburger, 2016]

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$A ::= x \; j \; x^{\perp} \; j \; A \quad A \; j \; 1 \; j \; A \quad A \; j \; ? \; j \; A \quad A \; j \; 0 \; j \; A \,\&\, A \; j \; > \; j \; !A \; j \; ?A$

$P \; ::= \quad j \; ! \; A$

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...not the same!

[Andreoli, 1990] [Laurent, 2004]

Modular focused systems for modal logics

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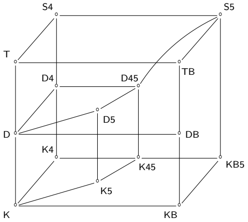
Classical normal modal logics:

k: $\Box(A \ ! \ B) \ ! \ (\Box A \ ! \ \Box B)$

d: $\Box A \ ! \ \Diamond A$ (Seriality)
t: $\Box A \ ! \ A$ (Reflexivity)
b: $\Diamond\Box A \ ! \ A$ (Symmetry)
4: $\Box A \ ! \ \Box\Box A$ (Transitivity)
5: $\Diamond\Box A \ ! \ \Box A$ (Euclideanness)



Modular focused systems for modal logics

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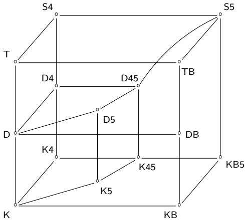

Nested sequent system:

Modular focused systems for modal logics

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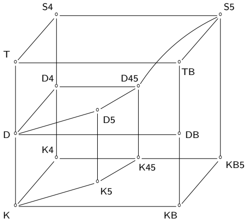

Nested sequent system:

1. complete and modular
   F is a theorem of K + axioms iff F is provable in KN + rules

[Brünnler, 2009]

Modular focused systems for modal logics

Classical normal modal logics:

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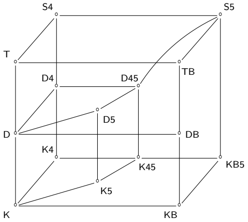


Nested sequent system:

1. complete and modular
   F is a theorem of K + axioms iff F is provable in KN + rules
   [Brünnler, 2009]
2. polarised and focused
   F theorem of K + axioms iff F has a focused proof in KN + rules
   [Chaudhuri, M., Strassburger, 2016]

Nested sequents

Nested sequents generalise sequents from a multiset of formulas

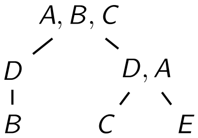
Sequent:

$$A, B, C$$

Nested sequents

Nested sequents generalise sequents from a multiset of formulas to a tree of multisets of formulas.

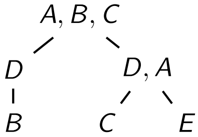
Nested sequent:



Nested sequents

In the sequent term, brackets indicate the parent-child relation in the tree

Nested sequent:

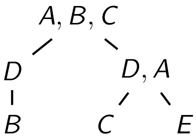


$$\Gamma = A, B, C, [D, [B]], [D, A, [C], [E]]$$

Nested sequents

In the sequent term, brackets indicate the parent-child relation in the tree and can be interpreted as the modal $\Box$.

Nested sequent:



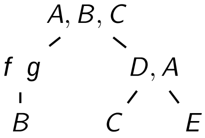
$$\Gamma = A, B, C, [D, [B]], [D, A, [C], [E]]$$

$$A \vee B \vee C \vee \Box(D \vee \Box B), \Box(D \vee A \vee \Box C \vee \Box E)$$

Nested sequents

A context is obtained by removing a formula and replacing it by a hole

Sequent context:

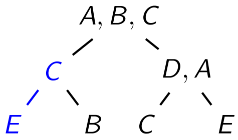


$$\Gamma f \;\; g = A, B, C, [f \;\; g, [B]], [D, A, [C], [E]]$$

Nested sequents

A context is obtained by removing a formula and replacing it by a hole that can then be filled by another nested sequent.

Sequent context:

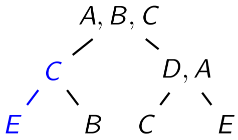


$$\Gamma f C, [E] g = A, B, C, [C, [E], [B]], [D, A, [C], [E]]$$

Nested sequents

This allows us to build rules than can be applied at any depth in the tree.

Sequent context:



$$\Gamma f C, [E] g = A, B, C, [C, [E], [B]], [D, A, [C], [E]]$$

The standard nested system for modal logics

Formulas: $A \quad ::= \quad x\ j\ x^{\perp}\ j\ A \wedge A\ j\ A \_\ A\ j\ \Box A\ j\ \Diamond A$

System KN**:**

$$\mathsf{c}\ \frac{\Gamma f A, A g}{\Gamma f A g} \qquad \Box\ \frac{\Gamma f [A] g}{\Gamma f \Box A g} \qquad \_\ \frac{\Gamma f A, B g}{\Gamma f A \_\ B g}$$

$$\mathsf{id}\ \frac{}{\Gamma f x^{\perp}, x g} \qquad \Diamond_{\mathsf{k}}\ \frac{\Gamma f [A, \Delta] g}{\Gamma f \Diamond A, [\Delta] g} \qquad \wedge\ \frac{\Gamma f A g \quad \Gamma f B g}{\Gamma f A \wedge B g}$$

The standard nested system for modal logics

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System KN:

$$\mathsf{c}\,\frac{\Gamma f A, A g}{\Gamma f A g} \qquad \Box\,\frac{\Gamma f [A] g}{\Gamma f \Box A g} \qquad \_\,\frac{\Gamma f A, B g}{\Gamma f A \_ \; B g}$$

$$\mathsf{id}\,\frac{}{\Gamma f x^{\perp}, x g} \qquad \Diamond_{\mathsf{k}}\,\frac{\Gamma f [A, \Delta] g}{\Gamma f \Diamond A, [\Delta] g} \qquad \wedge\,\frac{\Gamma f A g \quad \Gamma f B g}{\Gamma f A \wedge B g}$$

Modal rules:

$$\Diamond_{\mathsf{d}}\,\frac{\Gamma f [A] g}{\Gamma f \Diamond A g} \qquad \Diamond_{\mathsf{t}}\,\frac{\Gamma f A g}{\Gamma f \Diamond A g} \qquad \Diamond_{\mathsf{b}}\,\frac{\Gamma f [\Delta], A g}{\Gamma f [\Delta, \Diamond A] g} \qquad \Diamond_{4}\,\frac{\Gamma f [\Diamond A, \Delta] g}{\Gamma f \Diamond A, [\Delta] g} \qquad \Diamond_{5}\,\frac{\Gamma f \, ; g f \Diamond A g}{\Gamma f \Diamond A g f \, ; g}$$

d: $\Box A \; ! \; \Diamond A$ t: $A \; ! \; \Diamond A$ b: $A \; ! \; \Box \Diamond A$ 4: $\Diamond \Diamond A \; ! \; \Diamond A$ 5: $\Diamond A \; ! \; \Box \Diamond A$

[Brünnler, 2009]

The focused nested system for modal logics

Polarized formulas:

$$P \quad ::= \quad x \;\; j \; A \overset{+}{\wedge} A \; j \; A \overset{+}{\vee} A \; j \; \Diamond A$$

$$N \quad ::= \quad x^{\perp} j \; A \overset{-}{\vee} A \; j \; A \overset{-}{\wedge} A \; j \; \Box A$$

System KN**:**

$$\text{c}\ \frac{\Gamma f A, A g}{\Gamma f A g} \qquad \Box\ \frac{\Gamma f [A] g}{\Gamma f \Box A g} \qquad -\ \frac{\Gamma f A, B g}{\Gamma f A \_\ B g}$$

$$\text{id}\ \frac{}{\Gamma f x^{\perp}, x g} \qquad \Diamond_{\mathrm{k}}\ \frac{\Gamma f [A, \Delta] g}{\Gamma f \Diamond A, [\Delta] g} \qquad \wedge\ \frac{\Gamma f A g \quad \Gamma f B g}{\Gamma f A \wedge B g}$$

Modal rules:

$$\Diamond_{\mathrm{d}}\ \frac{\Gamma f [A] g}{\Gamma f \Diamond A g} \qquad \Diamond_{\mathrm{t}}\ \frac{\Gamma f A g}{\Gamma f \Diamond A g} \qquad \Diamond_{\mathrm{b}}\ \frac{\Gamma f [\Delta], A g}{\Gamma f [\Delta, \Diamond A] g} \qquad \Diamond_{4}\ \frac{\Gamma f [\Diamond A, \Delta] g}{\Gamma f \Diamond A, [\Delta] g} \qquad \Diamond_{5}\ \frac{\Gamma f ; g f \Diamond A g}{\Gamma f \Diamond A g f ; g}$$

The focused nested system for modal logics

Polarized formulas:

$$P \;::=\; x \mid A \overset{+}{\wedge} A \mid A \overset{+}{\vee} A \mid \Diamond A$$
$$N \;::=\; x^{\perp} \mid A \overset{-}{\vee} A \mid A \overset{-}{\wedge} A \mid \Box A$$

Focused system KNF**:**

$$\Box\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \overset{-}{\vee}\ \frac{\Gamma\{A, B\}}{\Gamma\{A \overset{-}{\vee} B\}} \qquad \overset{-}{\wedge}\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \overset{-}{\wedge} B\}}$$

$$\mathrm{id}\ \frac{}{\Gamma\{x^{\perp}, \langle x \rangle\}} \qquad \Diamond_k\ \frac{\Gamma\{[\langle A \rangle, \Delta]\}}{\Gamma\{\langle \Diamond A \rangle, [\Delta]\}} \qquad \wedge\ \frac{\Gamma\{\langle A \rangle\} \quad \Gamma\{\langle B \rangle\}}{\Gamma\{\langle A \overset{+}{\wedge} B \rangle\}} \qquad \overset{+}{\vee}_i\ \frac{\Gamma\{\langle A_i \rangle\}}{\Gamma\{\langle A_1 \overset{+}{\vee} A_2 \rangle\}}$$

Modal rules:

$$\Diamond_{\mathrm{d}}\ \frac{\Gamma\{[A]\}}{\Gamma\{\Diamond A\}} \qquad \Diamond_{\mathrm{t}}\ \frac{\Gamma\{A\}}{\Gamma\{\Diamond A\}} \qquad \Diamond_{\mathrm{b}}\ \frac{\Gamma\{[\Delta], A\}}{\Gamma\{[\Delta, \Diamond A]\}} \qquad \Diamond_{4}\ \frac{\Gamma\{[\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} \qquad \Diamond_{5}\ \frac{\Gamma\{\emptyset\}\{\Diamond A\}}{\Gamma\{\Diamond A\}\{\emptyset\}}$$

The focused nested system for modal logics

Polarized formulas:

$$P \quad ::= \quad x \ \ j\ A \overset{+}{\wedge} A\ j\ A \overset{+}{\vee} A\ j\ \diamond A$$
$$N \quad ::= \quad x^{\perp}\ j\ A \overset{-}{\vee} A\ j\ A \overset{-}{\wedge} A\ j\ \Box A$$

Focused system KNF**:**

$$\Box\ \frac{\Gamma f[A]g}{\Gamma f \Box A g} \qquad \overset{-}{\vee}\ \frac{\Gamma f A, B g}{\Gamma f A \overset{-}{\vee} B g} \qquad \overset{-}{\wedge}\ \frac{\Gamma f A g \quad \Gamma f B g}{\Gamma f A \overset{-}{\wedge} B g} \qquad \text{dec}\ \frac{\Gamma f P, h P i g}{\Gamma f P g}$$

$$\text{id}\ \frac{}{\Gamma f x^{\perp}, h x i g} \qquad \diamond_k\ \frac{\Gamma f [h A i, \Delta] g}{\Gamma f h \diamond A i, [\Delta] g} \qquad \wedge\ \frac{\Gamma f h A i g \quad \Gamma f h B i g}{\Gamma f h A \overset{+}{\wedge} B i g} \qquad \overset{+}{\vee}_i\ \frac{\Gamma f h A_i i g}{\Gamma f h A_1 \overset{+}{\vee} A_2 i g}$$

Modal rules:

$$\diamond_{\mathrm{d}}\ \frac{\Gamma f[A]g}{\Gamma f \diamond A g} \qquad \diamond_{\mathrm{t}}\ \frac{\Gamma f A g}{\Gamma f \diamond A g} \qquad \diamond_{\mathrm{b}}\ \frac{\Gamma f[\Delta], A g}{\Gamma f[\Delta, \diamond A] g} \qquad \diamond_4\ \frac{\Gamma f[\diamond A, \Delta] g}{\Gamma f \diamond A, [\Delta] g} \qquad \diamond_5\ \frac{\Gamma f ; g f \diamond A g}{\Gamma f \diamond A g f ; g}$$

The focused nested system for modal logics

Polarized formulas:

$$P \quad ::= \quad x \;\; j \; A \overset{+}{\wedge} A \; j \; A \overset{+}{\vee} A \; j \; \Diamond A$$

$$N \quad ::= \quad x^{\perp} \; j \; A \overset{-}{\vee} A \; j \; A \overset{-}{\wedge} A \; j \; \Box A$$

Focused system KNF**:**

$$\Box \; \frac{\Gamma f [A] g}{\Gamma f \Box A g} \qquad \overset{-}{\vee} \; \frac{\Gamma f A, B g}{\Gamma f A \overset{-}{\vee} B g} \qquad \overset{-}{\wedge} \; \frac{\Gamma f A g \quad \Gamma f B g}{\Gamma f A \overset{-}{\wedge} B g} \qquad \text{dec} \; \frac{\Gamma f P, h P i g}{\Gamma f P g}$$

$$\text{id} \; \frac{}{\Gamma f x^{\perp}, h x i g} \qquad \Diamond_k \; \frac{\Gamma f [h A i, \Delta] g}{\Gamma f h \Diamond A i, [\Delta] g} \qquad \wedge \; \frac{\Gamma f h A i g \quad \Gamma f h B i g}{\Gamma f h A \overset{+}{\wedge} B i g} \qquad \overset{+}{\vee}_i \; \frac{\Gamma f h A_i i g}{\Gamma f h A_1 \overset{+}{\vee} A_2 i g} \qquad \text{rel} \; \frac{\Gamma f N g}{\Gamma f h N i g}$$

Modal rules:

$$\Diamond_{\mathrm{d}} \; \frac{\Gamma f [A] g}{\Gamma f \Diamond A g} \qquad \Diamond_{\mathrm{t}} \; \frac{\Gamma f A g}{\Gamma f \Diamond A g} \qquad \Diamond_{\mathrm{b}} \; \frac{\Gamma f [\Delta], A g}{\Gamma f [\Delta, \Diamond A] g} \qquad \Diamond_4 \; \frac{\Gamma f [\Diamond A, \Delta] g}{\Gamma f \Diamond A, [\Delta] g} \qquad \Diamond_5 \; \frac{\Gamma f \, ; g f \Diamond A g}{\Gamma f \Diamond A g f \, ; g}$$

The focused nested system for modal logics

Polarized formulas:

$$P \quad ::= \quad x \;\; j \; A \overset{+}{\wedge} A \; j \; A \overset{+}{\vee} A \; j \; \Diamond A$$

$$N \quad ::= \quad x^{\perp} \; j \; A \overset{-}{\vee} A \; j \; A \overset{-}{\wedge} A \; j \; \Box A$$

Focused system KNF**:**

$$\Box \; \frac{\Gamma f [A] g}{\Gamma f \Box A g} \qquad \overset{-}{\vee} \; \frac{\Gamma f A, B g}{\Gamma f A \overset{-}{\vee} B g} \qquad \overset{-}{\wedge} \; \frac{\Gamma f A g \quad \Gamma f B g}{\Gamma f A \overset{-}{\wedge} B g} \qquad \text{dec} \; \frac{\Gamma f P, h P i g}{\Gamma f P g}$$

$$\text{id} \; \frac{}{\Gamma f x^{\perp}, h x i g} \qquad \Diamond_k \; \frac{\Gamma f [h A i, \Delta] g}{\Gamma f h \Diamond A i, [\Delta] g} \qquad \wedge \; \frac{\Gamma f h A i g \quad \Gamma f h B i g}{\Gamma f h A \overset{+}{\wedge} B i g} \qquad \overset{+}{\vee}_i \; \frac{\Gamma f h A_i i g}{\Gamma f h A_1 \overset{+}{\vee} A_2 i g} \qquad \text{rel} \; \frac{\Gamma f N g}{\Gamma f h N i g}$$

Focused modal rules:

$$\Diamond_d \; \frac{\Gamma f [h A i] g}{\Gamma f h \Diamond A i g} \qquad \Diamond_t \; \frac{\Gamma f h A i g}{\Gamma f h \Diamond A i g} \qquad \Diamond_b \; \frac{\Gamma f [\Delta], h A i g}{\Gamma f [\Delta, h \Diamond A i] g} \qquad \Diamond_4 \; \frac{\Gamma f [h \Diamond A i, \Delta] g}{\Gamma f h \Diamond A i, [\Delta] g} \qquad \Diamond_5 \; \frac{\Gamma f ; g f h \Diamond A i g}{\Gamma f h \Diamond A i g f ; g}$$

A nested system for MELL

Formulas: $A ::= x \; j \; x^{\perp} \; j \; A \quad A \; j \; 1 \; j \; A \quad A \; j \; ? \; j \; !A \; j \; ?A$

System NMELL**:**

$$\text{id}\ \frac{}{\Gamma^{[\,]} f x, x^{\perp} g} \qquad 1\ \frac{}{\Gamma^{[\,]} f 1 g}$$

$$?\ \frac{\Gamma f ; g}{\Gamma f ? g} \qquad \frac{\Gamma f A, B g}{\Gamma f A \quad B g} \qquad \frac{\Gamma f A g \quad \Delta f B g}{\Gamma \quad \Delta f A \quad B g} \qquad !\ \frac{\Gamma f [A] g}{\Gamma f !A g}$$

$$?_{\mathrm{t}}\ \frac{\Gamma f A g}{\Gamma f ?A g} \qquad ?_{4}\ \frac{\Gamma f [?A, \Delta] g}{\Gamma f ?A, [\Delta] g} \qquad ?_{\mathrm{c}}\ \frac{\Gamma f ?A, ?A g}{\Gamma f ?A g} \qquad ?_{\mathrm{w}}\ \frac{\Gamma f ; g}{\Gamma f ?A g}$$

1. $\Gamma^{[\,]} f \; g ::= f \; g \; j \; [\Gamma^{[\,]} f \; g]$
2. merge $\Gamma \quad \Delta f \; g$ when $depth(\Gamma f \; g) = depth(\Delta f \; g)$

A nested system for MELL

Exponentials:

$$? \; \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A} \qquad\qquad ?_{t} \; \frac{\Gamma\{A\}}{\Gamma\{?A\}}$$

A nested system for MELL

Exponentials:

$$?\ \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A} \qquad\qquad ?_t\ \frac{\Gamma\{A\}}{\Gamma\{?A\}}$$

$$!\ \frac{\vdash ?\Delta, A}{\vdash ?\Delta, !A} \qquad\qquad \begin{array}{c} \Gamma\{[?B_1, \ldots, ?B_n, A]\} \\ ?_4 \Big\| \\ ?_4\ \dfrac{\Gamma\{?B_1, \ldots, ?B_{n-1}, [?B_n, A]\}}{!\ \dfrac{\Gamma\{?B_1, \ldots, ?B_n, [A]\}}{\Gamma\{?B_1, \ldots, ?B_n, !A\}}} \end{array}$$

Could ! be negative like □?

Formulas: $A ::= x \mid x^{\perp} \mid A \quad A \mid 1 \mid A \quad A \mid ? \mid !A \mid ?A$

Could ! be negative like □?

Polarized formulas:

$$P \quad ::= \quad x \mid A \quad A \mid 1 \mid ?A$$
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Polarized formulas:

$$P \quad ::= \quad x \;\; j\; A \quad A\; j\; 1\; j\; ?A$$
$$N \quad ::= \quad x^{\perp} j\; A \quad A\; j\;\; ?\; j\; !A$$

A critical example:

$$\dfrac{\dfrac{1\;\dfrac{}{h1i} \qquad h?x^{\perp}i, !x \qquad !x}{h1 \qquad ?x^{\perp}i, !x \qquad !x}}{1 \qquad ?x^{\perp}, !x \qquad !x}\;\text{dec}$$

Could ! be negative like □?

Polarized formulas:

$$P \;::=\; x \;\; j\, A \quad A\, j\; 1\; j\; ?A$$
$$N \;::=\; x^{\perp} j\, A \quad A\, j\; ?\; j\; !A$$

A critical example:

$$\dfrac{\dfrac{\dfrac{}{h1i}\,1 \qquad h?x^{\perp}i, !x \qquad !x}{h1 \qquad ?x^{\perp}i, !x \qquad !x}}{1 \qquad ?x^{\perp}, !x \qquad !x}\,\text{dec}$$

$$\dfrac{\dfrac{\dfrac{}{1}\,1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[x^{\perp}, x]}\,\text{id}}{[?x^{\perp}, x]}\,?_t}{?x^{\perp}, [x]}\,?_4}{?x^{\perp}, !x}\,! \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{[x^{\perp}, x]}\,\text{id}}{[?x^{\perp}, x]}\,?_t}{?x^{\perp}, [x]}\,?_4}{?x^{\perp}, !x}\,!}{?x^{\perp}, ?x^{\perp}, !x \qquad !x}}{?x^{\perp}, !x \qquad !x}\,?_c}{1 \qquad ?x^{\perp}, !x \qquad !x}}{1 \qquad ?x^{\perp}, !x \qquad !x}$$

Conclusion

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Linear logic

$$\frac{}{\vdash \Gamma, a, \bar{a}} \qquad \frac{}{\vdash 1} \qquad \frac{}{\vdash \Gamma, \top} \qquad \frac{\Gamma}{\vdash \Gamma, \bot}$$

$$\frac{\vdash \Gamma_1, A \quad \vdash \Gamma_2, B}{\vdash \Gamma_1, \Gamma_2, A \quad B} \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \quad B} \quad \& \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B} \qquad {}_1\frac{\vdash \Gamma, A}{\vdash \Gamma, A \vee B} \qquad {}_2\frac{\vdash \Gamma, B}{\vdash \Gamma, A \vee B}$$

$$? \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A} \quad ! \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A} \quad \mathsf{c} \frac{\vdash \Gamma}{\vdash \Gamma, ?A} \quad \mathsf{w} \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}$$

The problem with adding additives

$$
\mathrm{!}\;\dfrac{\&\;\dfrac{{}^{1}\;\dfrac{?\;\dfrac{\mathrm{id}\;\dfrac{}{[?x^{\perp}, x]}}{?x^{\perp}, [x]}}{?x^{\perp} \quad ?x, [x]} \qquad {}^{2}\;\dfrac{?\;\dfrac{\mathrm{id}\;\dfrac{}{[?x, x^{\perp}]}}{?x, [x^{\perp}]}}{?x \quad ?x, [x^{\perp}]}}{?x^{\perp} \quad ?x, [x \mathbin{\&} x^{\perp}]}}{?x^{\perp} \quad ?x, !(x \mathbin{\&} x^{\perp})}
$$