Comparing □ and ! via polarities

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The answer

"from a proof-theoretical point of view exponentials behave exactly like S4 modalities"

[Martini & Masini, 1994]
"from a proof-theoretical point of view exponentials behave exactly like S4 modalities"

[Martini & Masini, 1994]
Wait...what? woo hoo.
Are ! and □ interchangeable?
Are $!$ and $\Box$ interchangeable?

**Modal logic S4:**

\[ A ::= x \mid x^\perp \mid A \land A \mid T \mid A \lor A \mid \bot \]
Are $!$ and $\square$ interchangeable?

**Modal logic** S4:

$$A ::= x \mid x^\perp \mid A \land A \mid \top \mid A \lor A \mid \bot \mid \Box A \mid \Diamond A$$
Are \( ! \) and \( \Box \) interchangeable?

**Modal logic S4:**

\[ A ::= x \mid x^\perp \mid A \land A \mid \top \mid A \lor A \mid \bot \mid \Box A \mid \Diamond A \]

**Linear logic LL:**

\[ A ::= x \mid x^\perp \mid A \otimes A \mid 1 \mid A \# A \mid \bot \mid A \oplus A \mid 0 \mid A \& A \mid \top \]
Are ! and □ interchangeable?

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**Modal logic S4:**

$$A ::= x \mid x^{\perp} \mid A \land A \mid \top \mid A \lor A \mid \bot \mid \Box A \mid \Diamond A$$

$$\frac{\vdash \Box \Gamma, A}{\vdash \Gamma, \Box A, \Delta} \quad \frac{\vdash \Gamma, \Box A, A}{\vdash \Gamma, \Diamond A}$$

**Linear logic LL:**

$$A ::= x \mid x^{\perp} \mid A \otimes A \mid 1 \mid A \lhd A \mid \bot \mid A \oplus A \mid 0 \mid A \& A \mid \top \mid !A \mid ?A$$

$$\frac{\vdash ? \Gamma, A}{\vdash ? \Gamma, !A} \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}$$
Are ! and □ interchangeable?

Modal logic S4:
\[ A ::= x \mid x ⊥ \mid A ∧ A \mid \top \mid A ∨ A \mid \bot \mid □A \mid ◇A \]

\[ \begin{align*}
□ & \vdash ◇Γ, □A, Δ \\
◇ & \vdash Γ, ◇A, A
\end{align*} \]

Linear logic LL:
\[ A ::= x \mid x ⊥ \mid A ⊗ A \mid 1 \mid A ⊸ A \mid \bot \mid A ⊕ A \mid 0 \mid A & A \mid \top \mid !A \mid ?A \]

\[ \begin{align*}
! & \vdash ?Γ, !A \\
? & \vdash Γ, ?A
\end{align*} \]
Are \( \Box \) and \( ! \) interchangeable?

**Theorem:** [Martini & Masini, 1994]

\[ \Gamma \text{ provable in } S4 \iff \Gamma^+ \text{ provable in } LL \]

**Modal logic S4:**

\[ A ::= x | x^\bot | A \land A | \top | A \lor A | \bot | \Box A | \Diamond A \]

\[
\begin{align*}
\Box & \vdash \Diamond \Gamma, A \\
\vdash \Diamond \Gamma, \Box A, \Delta \\
\Diamond & \vdash \Gamma, \Diamond A, A
\end{align*}
\]

**Linear logic LL:**

\[ A ::= x | x^\bot | A \otimes A | 1 | A \bowtie A | \bot | A \oplus A | 0 | A \& A | \top | !A | ?A \]

\[
\begin{align*}
! & \vdash ? \Gamma, A \\
\vdash ? \Gamma, !A \\
? & \vdash \Gamma, A \\
\vdash \Gamma, ?A
\end{align*}
\]
Are ! and □ interchangeable?

**Theorem:** [Martini & Masini, 1994]

\[ \Gamma \text{ provable in S4} \iff \Gamma^+ \text{ provable in LL} \]
Are ! and □ interchangeable?

**Theorem:** [Martini & Masini, 1994]

\[ \Gamma \text{ provable in S4} \iff \Gamma^+ \text{ provable in LL} \]

**Their answer:**

cut-free proof of an S4 sequent

\[ \uparrow \]

cut-free proof of its LL translation
Are ! and □ interchangeable?

**Theorem:** [Martini & Masini, 1994]

\[ \Gamma \text{ provable in S4} \iff \Gamma^+ \text{ provable in LL} \]

**Our question:** focused polarised cut-free proof of an S4 sequent

\[ \uparrow \]

focused polarised cut-free proof of its LL translation

?
Polarity and focusing

**Polarities:**
- **non-invertible rules**: positive connectives
- **invertible rules**: negative connectives

Inversion: $\pi \vdash N$, the last rule is negative.

Focus on a positive formula: $\pi \vdash P$, only rules decomposing $P$ between two rules decomposing $P$.

Completeness of focusing: if a formula $F$ is provable then $F$ has a focused proof.

[Andreoli, 1990] [Laurent, 2004]
Polarity and focusing

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Polarity and focusing

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**Modal logic S4:**
\[ A ::= x \ | \ x^\perp \ | \ A \land A \ | \ \top \ | \ A \lor A \ | \ \bot \ | \Box A \ | \Diamond A \]

**Linear logic LL:**
\[ A ::= x \ | \ x^\perp \ | \ A \otimes A \ | \ 1 \ | \ A \multimap A \ | \ \bot \ | \ A \oplus A \ | \ 0 \ | \ A \& A \ | \ \top \ | \ !A \ | \ ?A \]

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\[ A ::= x \mid x^\bot \mid A \otimes A \mid 1 \mid A \bowtie A \mid \bot \mid A \oplus A \mid 0 \mid A \& A \mid \top \mid !A \mid ?A \]
\[ P ::= x \mid A \otimes A \mid 1 \mid A \oplus A \mid 0 \]
\[ N ::= x^\bot \mid A \bowtie A \mid \bot \mid A \& A \mid \top \]

[Andreoli, 1990] [Laurent, 2004]
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Modal logic S4:
\[ A ::= x \mid x^\bot \mid A \land A \mid \top \mid A \lor A \mid \bot \mid \Box A \mid \Diamond A \]
\[ P ::= x \mid A \overset{\#}{\land} A \mid \top \mid A \overset{\dagger}{\lor} A \mid \bot \]
\[ N ::= x^\bot \mid A \overset{\dagger}{\lor} A \mid \bot \mid A \overset{\top}{\land} A \mid \top \]

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\[ A ::= x \mid x^\bot \mid A \otimes A \mid 1 \mid A \multimap A \mid \bot \mid A \oplus A \mid 0 \mid A \& A \mid \top \mid !A \mid ?A \]
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[Miller, Volpe, 2015] [Chaudhuri, M., Strassburger, 2016]

Linear logic LL:
\[ A ::= x \mid x^\bot \mid A \otimes A \mid 1 \mid A \oplus A \mid \bot \mid A \land A \mid 0 \mid A \& A \mid \top \mid \! A \mid ? A \]

\[ P ::= x \mid A \otimes A \mid 1 \mid A \oplus A \mid 0 \mid \! A \]
\[ N ::= x^\bot \mid A \oplus A \mid \bot \mid A \land A \mid \top \mid ? A \]

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Polarity and connectives

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non-invertible rules : positive connectives  
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Modal logic S4:  
$A ::= x \mid x^\perp \mid A \land A \mid \top \mid A \lor A \mid \bot \mid \Box A \mid \Diamond A$

$P ::= \diamond A \quad \text{This is...}$

$N ::= \Box A$

[Miller, Volpe, 2015] [Chaudhuri, M., Strassburger, 2016]

Linear logic LL:  
$A ::= x \mid x^\perp$
Modular focused systems for modal logics

Classical normal modal logics:

- **k**: $(A \rightarrow B) \rightarrow (2A \rightarrow 2B)$
- **d**: $2A \rightarrow 3A$ (Seriality)
- **t**: $2A \rightarrow A$ (Reflexivity)
- **b**: $3A \rightarrow A$ (Symmetry)
- **4**: $2A \rightarrow 22A$ (Transitivity)
- **5**: $3A \rightarrow 2A$ (Euclideanness)

Nested sequent system:

1. complete and modular
   \[ F \text{ is a theorem of } K + \text{axioms} \iff F \text{ is provable in } KN + \text{rules} \]
   \[ \text{[Br"unnler, 2009]} \]
2. polarised and focused
   \[ F \text{ theorem of } K + \text{axioms} \iff F \text{ has a focused proof in } KN + \text{rules} \]
   \[ \text{[Chaudhuri, M., Strassburger, 2016]} \]
Modular focused systems for modal logics

Classical normal modal logics:

k: \(\Box (A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)\)

d: \(\Box A \rightarrow \Diamond A\) (Seriality)

t: \(\Box A \rightarrow A\) (Reflexivity)

b: \(\Diamond \Box A \rightarrow A\) (Symmetry)

4: \(\Box A \rightarrow \Box \Box A\) (Transitivity)

5: \(\Diamond \Box A \rightarrow \Box A\) (Euclideanness)
Modular focused systems for modal logics

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2. polarised and focused
   $F$ theorem of $K +$ axioms iff $F$ has a focused proof in $KN +$ rules
   [Chaudhuri, M., Strassburger, 2016]
Nested sequents generalise sequents from a multiset of formulas

Sequent:

\[ A, B, C \]
Nested sequents

Nested sequents generalise sequents from a multiset of formulas to a tree of multisets of formulas.

**Nested sequent:**

\[
\begin{array}{c}
A, B, C \\
\begin{array}{c}
D \\
\end{array} & \\
\begin{array}{c}
D, A \\
\end{array} \\
B & C & E
\end{array}
\]
Nested sequents

In the sequent term, brackets indicate the parent-child relation in the tree

Nested sequent:

\[ \Gamma = A, B, C, [D, [B]], [D, A, [C], [E]] \]
In the sequent term, brackets indicate the parent-child relation in the tree and can be interpreted as the modal $\square$.

**Nested sequent:**

\[
\begin{array}{c}
A, B, C \\
\quad \quad \quad \quad \quad D, A \\
\quad \quad \quad \quad B \quad C \quad E
\end{array}
\]

\[\Gamma = A, B, C, [D, [B]], [D, A, [C], [E]]\]

\[A \lor B \lor C \lor \square(D \lor \square B), \square(D \lor A \lor \square C \lor \square E)\]
Nested sequents

A context is obtained by removing a formula and replacing it by a hole

Sequent context:

\[ \Gamma \{ \} = A, B, C, \{ \}, [B], [D, A, [C], [E]] \]
A context is obtained by removing a formula and replacing it by a hole that can then be filled by another nested sequent.

**Sequent context:**

\[
\Gamma \{C, [E]\} = A, B, C, [C, [E], [B]], [D, A, [C], [E]]
\]
Nested sequents

This allows us to build rules that can be applied at any depth in the tree.

**Sequent context:**

\[ \Gamma \{ C, [E] \} = A, B, C, [C, [E], [B]], [D, A, [C], [E]] \]
The standard nested system for modal logics

Formulas: \[ A ::= x \mid x^\perp \mid A \land A \mid A \lor A \mid \Box A \mid \Diamond A \]

System KN:

\[
\begin{align*}
&c \quad \frac{\Gamma\{A, A\}}{\Gamma\{A\}} \\
&\Box \quad \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \\
&\lor \quad \frac{\Gamma\{A, B\}}{\Gamma\{A \lor B\}} \\
&id \quad \frac{\Gamma\{x^\perp, x\}}{\Gamma\{x^\perp, x\}} \\
&\Diamond_k \quad \frac{\Gamma\{[A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} \\
&\land \quad \frac{\Gamma\{A\} \Gamma\{B\}}{\Gamma\{A \land B\}}
\end{align*}
\]

[Brünnler, 2009]
The standard nested system for modal logics

**Formulas:**

\[ A ::= x \mid x^\perp \mid A \land A \mid A \lor A \mid \square A \mid \diamond A \]

**System KN:**

\[
\begin{align*}
\Gamma \{A, A\} & \quad \square \Gamma \{[A]\} & \quad \lor \Gamma \{A, B\} \\
\Gamma \{\} & \quad \Gamma \{A\} & \quad \Gamma \{A \lor B\} \\
\end{align*}
\]

\[
\begin{align*}
\Gamma \{x^\perp, x\} & \quad \diamond \Gamma \{A, \Delta\} & \quad \land \Gamma \{A\} \Gamma \{B\} \\
\Gamma \{\} & \quad \Gamma \{\diamond A, [\Delta]\} & \quad \land \Gamma \{A \land B\} \\
\end{align*}
\]

**Modal rules:**

\[
\begin{align*}
\diamond d & : \Gamma \{[A]\} \quad \diamond t & : \Gamma \{A\} \\
\diamond b & : \Gamma \{[\Delta], A\} & \diamond 4 & : \Gamma \{[\diamond A, \Delta]\} \\
\diamond 5 & : \Gamma \{\emptyset\} \Gamma \{\diamond A\} \\
\end{align*}
\]

\[
\begin{align*}
d: \square A & \rightarrow \diamond A & t: A & \rightarrow \diamond A & b: A & \rightarrow \square \diamond A \\
4: \diamond \diamond A & \rightarrow \diamond A & 5: \diamond A & \rightarrow \square \diamond A \\
\end{align*}
\]

[Brünnler, 2009]
The focused nested system for modal logics

Polarized formulas: $P ::= x | A \uparrow A | A \downarrow A | \Diamond A$

$N ::= x^\perp | A \lor A | A \land A | \Box A$

System KN:

- $\Gamma \{A, A\} \quad \Box \Gamma \{[A]\}$
- $\Gamma \{[A]\} \quad \lor \Gamma \{A \lor B\}$
- $\Gamma \{x^\perp, x\} \quad \Diamond_k \Gamma \{\Diamond A, [\Delta]\}$
- $\Gamma \{A\} \quad \Gamma \{B\}$
- $\Gamma \{A \land B\}$

Modal rules:

- $\Diamond \Gamma \{[A]\}$
- $\Diamond \Gamma \{A\}$
- $\Diamond \Gamma \{[\Delta], A\}$
- $\Diamond \Gamma \{[\Diamond A, [\Delta]\}$
- $\Diamond \Gamma \{\emptyset\}$
The focused nested system for modal logics

Polarized formulas:

\[ P ::= x | A \uparrow A | A \downarrow A | \Diamond A \]

\[ N ::= x \perp | A \triangledown A | A \triangleleft A | \Box A \]

Focused system \text{KNF}:

\[
\begin{array}{cccc}
\Box & \frac{\Gamma \{ [A] \}}{\Gamma \{ \Box A \}} & \checkmark & \frac{\Gamma \{ A, B \}}{\Gamma \{ A \triangledown B \}} & \checkmark & \frac{\Gamma \{ A \} \quad \Gamma \{ B \}}{\Gamma \{ A \triangleleft B \}} \\
\text{id} & \frac{\Gamma \{ x \perp, \langle x \rangle \}}{\Gamma \{ \langle A \rangle, \Delta \}} & \Diamond & \frac{\Gamma \{ \langle A \rangle \} \quad \Gamma \{ \langle B \rangle \}}{\Gamma \{ \langle A \uparrow B \rangle \}} & \checkmark & \frac{\Gamma \{ \langle A_i \rangle \}}{\Gamma \{ \langle A_1 \uparrow A_2 \rangle \}} \\
\end{array}
\]

Modal rules:

\[
\begin{array}{cccc}
\Diamond & \frac{\Gamma \{ [A] \}}{\Gamma \{ \Diamond A \}} & \Diamond & \frac{\Gamma \{ A \}}{\Gamma \{ \Diamond A \}} & \Diamond & \frac{\Gamma \{ [\Delta], A \}}{\Gamma \{ [\Delta, \Diamond A] \}} & \Diamond & \frac{\Gamma \{ [\Diamond A, \Delta] \}}{\Gamma \{ [\Diamond A, [\Delta] \}} & \Diamond & \frac{\Gamma \{ \emptyset \} \{ \Diamond A \}}{\Gamma \{ \Diamond A \} \{ \emptyset \}} \\
\end{array}
\]
The focused nested system for modal logics

Polarized formulas:
\[ P ::= x | A \uparrow A | A \downarrow A | \Diamond A \]
\[ N ::= x \perp | A \lor A | A \land A | \Box A \]

Focused system $\text{KNF}$:
\[
\begin{align*}
\Box \frac{\Gamma \{ [A] \}}{\Gamma \{ \Box A \}} & & \lor \frac{\Gamma \{ A, B \}}{\Gamma \{ A \lor B \}} & & \land \frac{\Gamma \{ A \} \Gamma \{ B \}}{\Gamma \{ A \land B \}} \quad \text{dec} \frac{\Gamma \{ P, \langle P \rangle \}}{\Gamma \{ P \}} \\
\text{id} \frac{\Gamma \{ x \perp, \langle x \rangle \}}{\Gamma \{ x \perp, \langle x \rangle \}} & & \Diamond \frac{\Gamma \{ \langle A \rangle, \Delta \} }{\Gamma \{ \Diamond A, [\Delta] \}} & & \frac{\Gamma \{ \langle A \rangle \} \Gamma \{ \langle B \rangle \} }{\Gamma \{ \langle A \land B \rangle \} } & & \lor_i \frac{\Gamma \{ \langle A_i \rangle \} }{\Gamma \{ \langle A_1 \lor A_2 \rangle \} }
\end{align*}
\]

Modal rules:
\[
\begin{align*}
\Diamond \text{d} \frac{\Gamma \{ [A] \} }{\Gamma \{ \Diamond A \} } & & \Diamond \text{t} \frac{\Gamma \{ A \} }{\Gamma \{ \Diamond A \} } & & \Diamond \text{b} \frac{\Gamma \{ [\Delta], A \} }{\Gamma \{ [\Delta], \Diamond A \} } & & \Diamond \text{i} \frac{\Gamma \{ \Diamond A, [\Delta] \} }{\Gamma \{ \Diamond A, \Delta \} } & & \Diamond \text{5} \frac{\Gamma \{ \emptyset \} \{ \Diamond A \} }{\Gamma \{ \Diamond A \} \{ \emptyset \} }
\end{align*}
\]
The focused nested system for modal logics

Polarized formulas:

\[ P ::= x \mid A \uparrow A \mid A \downarrow A \mid \lozenge A \]
\[ N ::= x \perp \mid A \lor A \mid A \land A \mid \Box A \]

Focused system KNF:

\[
\begin{align*}
\Box & \frac{\Gamma \{ [A] \}}{\Gamma \{ \Box A \}} \\
\lor & \frac{\Gamma \{ A, B \}}{\Gamma \{ A \lor B \}} \\
\land & \frac{\Gamma \{ A \} \Gamma \{ B \}}{\Gamma \{ A \land B \}} \\
dec & \frac{\Gamma \{ P, \langle P \rangle \}}{\Gamma \{ P \}}
\end{align*}
\]

id \[ \frac{\Gamma \{ x \perp, \langle \rangle \} \quad \Box_k}{\Gamma \{ \langle \Box A \rangle, [\Delta] \}} \]
\[ \Rightarrow \frac{\Gamma \{ [\langle \rangle, \Delta] \}}{\Gamma \{ \langle \rangle \}} \]
\[ \land \frac{\Gamma \{ \langle A \rangle \} \Gamma \{ \langle B \rangle \}}{\Gamma \{ \langle A \land B \rangle \}} \]
\[ \lor \frac{\Gamma \{ \langle A \rangle \} \Gamma \{ \langle B \rangle \}}{\Gamma \{ \langle A \lor B \rangle \}} \]
\[ \Rightarrow \frac{\Gamma \{ \langle A \rangle \} \Gamma \{ \langle B \rangle \}}{\Gamma \{ \langle A \land B \rangle \}} \]
\[ \Rightarrow \frac{\Gamma \{ \langle A \rangle \} \Gamma \{ \langle B \rangle \}}{\Gamma \{ \langle A \lor B \rangle \}} \]
\[ \Box_k \frac{\Gamma \{ [\langle \rangle, \Delta] \}}{\Gamma \{ [\langle \rangle, \Delta] \}} \]
\[ \Rightarrow \frac{\Gamma \{ \langle A \rangle \} \Gamma \{ \langle B \rangle \}}{\Gamma \{ \langle A \land B \rangle \}} \]
\[ \Rightarrow \frac{\Gamma \{ \langle A \rangle \} \Gamma \{ \langle B \rangle \}}{\Gamma \{ \langle A \lor B \rangle \}} \]
\[ \Rightarrow \frac{\Gamma \{ \langle A \rangle \} \Gamma \{ \langle B \rangle \}}{\Gamma \{ \langle A \land B \rangle \}} \]
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\[ \Rightarrow \frac{\Gamma \{ \langle A \rangle \} \Gamma \{ \langle B \rangle \}}{\Gamma \{ \langle A \lor B \rangle \}} \]
\[ \Rightarrow \frac{\Gamma \{ \langle A \rangle \} \Gamma \{ \langle B \rangle \}}{\Gamma \{ \langle A \land B \rangle \}} \]
\[ \Box_k \frac{\Gamma \{ [\langle \rangle, \Delta] \}}{\Gamma \{ [\langle \rangle, \Delta] \}} \]

Modal rules:

\[
\begin{align*}
\Box_d & \frac{\Gamma \{ [A] \}}{\Gamma \{ \Box A \}} \\
\Box_t & \frac{\Gamma \{ A \}}{\Gamma \{ \Box A \}} \\
\Box_b & \frac{\Gamma \{ [\Delta], A \}}{\Gamma \{ \Box A, [\Delta] \}} \\
\Box_4 & \frac{\Gamma \{ [\lozenge A, \Delta] \}}{\Gamma \{ \Box A, [\Delta] \}} \\
\Box_5 & \frac{\Gamma \{ \emptyset \} \Gamma \{ \lozenge A \}}{\Gamma \{ \Box A \} \Gamma \{ \emptyset \}}
\end{align*}
\]
The focused nested system for modal logics

Polarized formulas:

\[ P ::= x \mid A \top A \mid A \bot A \mid \lozenge A \]

\[ N ::= x \bot \mid A \top A \mid A \bot A \mid \Box A \]

Focused system **KNF**:

\[ \square \frac{\Gamma \{A\}}{\Gamma \{\Box A\}} \]

\[ \Downarrow \frac{\Gamma \{A, B\}}{\Gamma \{A \top B\}} \]

\[ \lozenge \frac{\Gamma \{A\} \quad \Gamma \{B\}}{\Gamma \{A \bot B\}} \]

[dec] \quad \frac{\Gamma \{P, \langle P\rangle\}}{\Gamma \{P\}}

\[ \text{id} \quad \frac{\Gamma \{x \bot, \langle x\rangle\}}{\Gamma \{\langle x\rangle, [\Delta]\}} \]

\[ k \quad \frac{\Gamma \{\langle A\rangle, [\Delta]\}}{\Gamma \{\langle \Diamond A\rangle\}} \]

\[ \wedge \quad \frac{\Gamma \{\langle A\rangle\} \quad \Gamma \{\langle B\rangle\}}{\Gamma \{\langle A \top B\rangle\}} \]

\[ \Downarrow \quad \frac{\Gamma \{\langle A_i\rangle\}}{\Gamma \{\langle A_1 \top A_2\rangle\}} \]

[rel] \quad \frac{\Gamma \{N\}}{\Gamma \{\langle N\rangle\}}

Focused modal rules:

\[ d \quad \frac{\Gamma \{\langle A\rangle\}}{\Gamma \{\langle \Diamond A\rangle\}} \]

\[ t \quad \frac{\Gamma \{\langle A\rangle\}}{\Gamma \{\langle \Diamond A\rangle\}} \]

\[ b \quad \frac{\Gamma \{[\Delta], \langle A\rangle\}}{\Gamma \{[\Delta], \langle \Diamond A\rangle\}} \]

\[ 4 \quad \frac{\Gamma \{\langle \Diamond A\rangle, \langle A\rangle\}}{\Gamma \{\langle \Diamond A\rangle, [\Delta]\}} \]

\[ 5 \quad \frac{\Gamma \{\langle A\rangle\}}{\Gamma \{\langle \Diamond A\rangle\}} \]
A nested system for MELL

Formulas: \( A ::= x \mid x^⊥ \mid A \otimes A \mid 1 \mid A \mathcal{R} A \mid \bot \mid !A \mid ?A \)

System NMELL:

1. \( \Gamma[\{\}]:= \{\} \mid [\Gamma[\{\}]] \)
2. merge \( \Gamma \cdot \Delta \{\} \) when \( \text{depth}(\Gamma\{\}) = \text{depth}(\Delta\{\}) \)
A nested system for MELL

**Exponentials:**

\[
\begin{align*}
? & \quad \vdash \Gamma, A \\
\vdash & \quad \Gamma, ?A \\
\end{align*}
\]

\[
\begin{align*}
? & \quad \frac{\Gamma\{A\}}{} \\
\vdash & \quad \Gamma\{?A\} \\
\end{align*}
\]
A nested system for MELL

Exponentials:

\[ \vdash \Gamma, A \]
\[ \vdash \Gamma, ?A \]

\[ \vdash ?\Delta, A \]
\[ \vdash ?\Delta, !A \]

\[ \Gamma\{\{A\}\} \]
\[ \Gamma\{?A\} \]

\[ \Gamma\{[?B_1, \ldots, ?B_n, A]\} \]
\[ ?_4 \parallel \]

\[ \Gamma\{[?B_1, \ldots, ?B_{n-1}, [?B_n, A]\} \]
\[ ! \frac{\Gamma\{[?B_1, \ldots, ?B_n, A]\}}{\Gamma\{[?B_1, \ldots, ?B_n, !A]\}} \]
Could \(!\) be negative like \(\square\)?

**Formulas:**

\[
A ::= x \mid x^{\perp} \mid A \otimes A \mid 1 \mid A \boxtimes A \mid \bot \mid !A \mid ?A
\]
Could ! be negative like □?

Polarized formulas:

\[
P ::= x \mid A \otimes A \mid 1 \mid ?A
\]

\[
N ::= x^\perp \mid A \mathbin{\&\&} A \mid \bot \mid !A
\]
Could ! be negative like □?

Polarized formulas:

\[ P ::= x | A \otimes A | 1 | ?A \]
\[ N ::= x^\perp | A \& A | \bot | !A \]

A critical example:

\[
\begin{align*}
1 & \quad \langle 1 \rangle \langle ?x^\perp \rangle, !x \otimes !x \\
\otimes & \quad \langle 1 \otimes ?x^\perp \rangle, !x \otimes !x \\
\text{dec} & \quad 1 \otimes ?x^\perp, !x \otimes !x
\end{align*}
\]
Could \(!\) be negative like \(\Box\)?

Polarized formulas:  
\[
P ::= x \mid A \otimes A \mid 1 \mid ?A
\]
\[
N ::= x^\perp \mid A \# A \mid \perp \mid !A
\]

A critical example:

\[
\begin{array}{c}
1 \\
\otimes
\end{array}
\frac{
\langle 1 \rangle \langle ? x^\perp \rangle, ! x \otimes ! x
}{
\langle 1 \otimes ? x^\perp \rangle, ! x \otimes ! x
}\]
\]
\[
\begin{array}{c}
\text{dec}
\end{array}
\frac{
1 \\
\otimes
\end{array}
\frac{
1 \otimes ? x^\perp, ! x \otimes ! x
}{
1 \otimes ? x^\perp, ! x \otimes ! x
}\]
\]

\[
\begin{array}{c}
id
\end{array}
\frac{
[x^\perp, x]
}{
[x^\perp, x]
}\]
\]

\[
\begin{array}{c}
? t
\end{array}
\frac{
[x^\perp, x]
}{
[x^\perp, x]
}\]
\]

\[
\begin{array}{c}
? 4
\end{array}
\frac{
[x^\perp, x]
}{
[x^\perp, x]
}\]
\]

\[
\begin{array}{c}
? c
\end{array}
\frac{
[x^\perp, x^\perp, ! x \otimes ! x]
}{
[x^\perp, x^\perp, ! x \otimes ! x]
}\]
\]
Exponentials do not behave like S4 modalities in terms of polarities.
Exponentials do not behave like S4 modalities in terms of polarities in multiplicative linear logic.
Exponentials do not behave like S4 modalities in terms of polarities in *multiplicative linear logic*.

It actually does not seem to come from the depth of the formalism.
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Exponentials do not behave like S4 modalities in terms of polarities in *multiplicative linear logic*. It actually does not seem to come from the depth of the formalism but from the interaction between exponentials and the other connectives. What about smaller fragments?
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What about smaller fragments? Tensorial logic?
Conclusion

Exponentials do not behave like S4 modalities in terms of polarities in *multiplicative linear logic*.

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What about smaller fragments? Tensorial logic?
Insights from category theory?
Conclusion

Exponentials do not behave like S4 modalities in terms of polarities in **multiplicative linear logic**.

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What about smaller fragments? Tensorial logic?
Insights from category theory?
Other comments?
Conclusion

Exponentials do not behave like S4 modalities in terms of polarities in multiplicative linear logic.

It actually does not seem to come from the depth of the formalism but from the interaction between exponentials and the other connectives.

What about smaller fragments? Tensorial logic?

Insights from category theory?

Other comments?
Linear logic

\[ \Gamma \vdash \Gamma, a, \bar{a} \quad \vdash 1 \quad \vdash \Gamma, \top \quad \vdash \Gamma, \bot \]

\[ \begin{align*}
\Gamma_1, A &\vdash \Gamma_2, B \\
\otimes &\vdash \Gamma_1, \Gamma_2, A \otimes B \\
\Gamma &\vdash \Gamma, A \otimes B \\
\& &\vdash \Gamma, A \otimes B \\
\& &\vdash \Gamma, A \otimes B \\
\oplus_1 &\vdash \Gamma, A \otimes B \\
\oplus_2 &\vdash \Gamma, A \otimes B \\
\Gamma &\vdash \Gamma, A \otimes B \\
\Gamma &\vdash \Gamma, A \otimes B \\
\Gamma &\vdash \Gamma, A \otimes B \\
\Gamma &\vdash \Gamma, A \otimes B \\
\Gamma &\vdash \Gamma, A \otimes B \\
? &\vdash \Gamma, ?A \\
? &\vdash \Gamma, ?A \\
? &\vdash \Gamma, !A \\
? &\vdash \Gamma, !A \\
? &\vdash \Gamma, ?A \\
? &\vdash \Gamma, ?A \\
? &\vdash \Gamma, ?A \\
? &\vdash \Gamma, ?A \\
? &\vdash \Gamma, ?A \\
? &\vdash \Gamma, ?A \\
? &\vdash \Gamma, ?A \\
? &\vdash \Gamma, ?A \\
\end{align*} \]
The problem with adding additives

\[
\begin{align*}
\text{id} & \quad \frac{[?x^\perp, x]}{?x^\perp, [x]} & \quad \text{id} & \quad \frac{[?x, x^\perp]}{?x, [x^\perp]} \\
\oplus_1 & \quad \frac{?x^\perp \oplus ?x, [x]}{\text{id} \quad \frac{[?x^\perp, x]}{?x^\perp, [x]}} & \quad \oplus_2 & \quad \frac{?x \oplus ?x, [x^\perp]}{\text{id} \quad \frac{[?x, x^\perp]}{?x, [x^\perp]}} \\
\& \quad \frac{?x^\perp \oplus ?x, [x \& x^\perp]}{\text{id} \quad \frac{[?x^\perp, x]}{?x^\perp, [x]}} & \quad \& \quad \frac{?x \oplus ?x, [x^\perp]}{\text{id} \quad \frac{[?x, x^\perp]}{?x, [x^\perp]}} \\
\& \quad \frac{?x^\perp \oplus ?x, !(x \& x^\perp)}{\text{id} \quad \frac{[?x^\perp, x]}{?x^\perp, [x]}} & \quad \& \quad \frac{?x \oplus ?x, [x^\perp]}{\text{id} \quad \frac{[?x, x^\perp]}{?x, [x^\perp]}}
\end{align*}
\]