Justification logic for constructive modal logic

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The big picture

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Justification logic:
Gödel:
What is the classical provability semantics of intuitionistic logic?
Artemov:
Logic of Proofs gives an operational view of this S4 type of provability.

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Extensions: realisation of logics below and above S4

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Intuitionistic variants: Some investigations toward

- realisation theorems (Artemov/Steren and Bonelli),
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but where the modal language is restricted to the $\Box$ modality.

However, intuitionistically $\Diamond$ cannot simply be viewed as the dual of $\Box$.

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We start with Artemov's treatment of the $\Box$-fragment of intuitonistic modal logic.

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Intuitionistic modal logic?
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The focus: on constructive versions of modal logic.

Constructive modal logic

Formulas: $A ::= \bot \mid a \mid A \wedge A \mid A \vee A \mid A \supset A$

Logic CK**:** Intuitionistic Propositional Logic

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Logic CK**:** Intuitionistic Propositional Logic

$$+ \quad \begin{array}{ll} \mathsf{k}_1: & \Box(A \supset B) \supset (\Box A \supset \Box B) \\ \mathsf{k}_2: & \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) \end{array} \qquad + \quad \text{necessitation:} \quad \frac{A}{\Box A}$$

(Wijesekera/Bierman and de Paiva/Mendler and Scheele)

Justification logic

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In the constructive version, we also add witness terms into the language.

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Justification logic

Modal formulas: $A ::= \bot \mid a \mid A \wedge A \mid A \vee A \mid A \supset A \mid \Box A$

Justification formulas: $A ::= \bot \mid a \mid A \wedge A \mid A \vee A \mid A \supset A \mid t : A$

Grammar of terms:

$$t ::= \quad c \quad \mid \quad x \quad \mid \quad (t \cdot t) \quad \mid \quad (t + t) \quad \mid \quad !\,t$$

Justification logic for constructive modal logic

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$!$: *proof checker*

α : witness variables

$\star$: execution

$\sqcup$: disjoint witness union

Justification logic for constructive modal logic

Axiomatisation JCK**:**

taut: Complete finite set of axioms for intuitionistic propositional logic

$\mathsf{jk}_\Box$: $t : (A \supset B) \supset (s : A \supset t \cdot s : B)$

sum: $s : A \supset (s + t) : A$ and $t : A \supset (s + t) : A$

$$\text{mp}\ \frac{A \supset B \quad A}{B} \qquad\qquad \text{ian}\ \frac{A \text{ is an axiom instance}}{c_1 : \ldots c_n : A}$$

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The machinery

Application: $\mathsf{jk}_{\square} : t : (A \supset B) \supset (s : A \supset t \cdot s : B)$

If t is a proof of $A \supset B$ and s is a proof of A, then $t \cdot s$ is a proof of B.

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Witness execution: $\mathsf{jk}_\Diamond : t : (A \supset B) \supset (\mu : A \supset t \star \mu : B)$

If t is a proof of $A \supset B$ and μ is a witness for A, then the same model denoted $t \star \mu$ is also a witness for B.

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If t is a proof of $A \supset B$ and μ is a witness for A, then the same model denoted $t \star \mu$ is also a witness for B.

Sum and union: $s:A \supset (s+t):A$, $\mu:A \supset (\mu \sqcup \nu):B, \ldots$
We adopt Artemov's $+$ to incorporate monotonicity of reasoning, and also transpose it on the witness side with $\sqcup$.

Iterated axiom necessitation and modus ponens:

The machinery

Justification logic can internalise its own reasoning.

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Lifting Lemma:

- If $A_1, \ldots, A_n \vdash_{\mathsf{JCK}} B$, then there exists a proof term $t(x_1, \ldots, x_n)$ such that, for all terms $s_1, \ldots, s_n$

$$\vdash_{\mathsf{JCK}} s_1 : A_1 \wedge \ldots \wedge s_n : A_n \supset t(s_1, \ldots, s_n) : B$$

- If $A_1, \ldots, A_n, C \vdash_{\mathsf{JCK}} B$, then there exists a witness term $\mu(x_1, \ldots, x_n, \beta)$ such that, for all terms $s_1, \ldots, s_n$ and ν

$$\vdash_{\mathsf{JCK}} s_1 : A_1 \wedge \ldots \wedge s_n : A_n \wedge \nu : C \supset \mu(s_1, \ldots, s_n, \nu) : B$$

Correspondence

Forgetful projection: If $\vdash_{\mathsf{JCK}} F$, then $\vdash_{\mathsf{CK}} F^\circ$,

where $(\cdot)^\circ$ maps justification formulas onto modal formulas, in particular:

$$(t : A)^\circ := \Box A^\circ \qquad\qquad (\mu : A)^\circ := \Diamond A^\circ$$

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Standard solution:
Consider a proof of the modal theorem in a cut-free sequent calculus.

Sequent calculus for modal logic

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Sequent system LCK**:**

$$\mathsf{id}\ \frac{}{\Gamma, a \Rightarrow a} \qquad \bot_{\mathsf{L}}\ \frac{}{\Gamma, \bot \Rightarrow C}$$

$$\vee_{\mathsf{L}}\ \frac{\Gamma, A \Rightarrow C \quad \Gamma, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C} \qquad \vee_{\mathsf{R}}\ \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B} \qquad \vee_{\mathsf{R}}\ \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}$$

$$\wedge_{\mathsf{L}}\ \frac{\Gamma, A, B \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C} \qquad \wedge_{\mathsf{R}}\ \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$\supset_{\mathsf{L}}\ \frac{\Gamma, A \supset B \Rightarrow A \quad \Gamma, B \Rightarrow C}{\Gamma, A \supset B \Rightarrow C} \qquad \supset_{\mathsf{R}}\ \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B}$$

Sequent calculus for modal logic

Sequent system LCK: $A_1, \ldots, A_n \Rightarrow C \quad \rightsquigarrow \quad (A_1 \wedge \ldots \wedge A_n) \supset C$

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$$\mathsf{k}_\Box\ \frac{\Gamma \Rightarrow A}{\Box\Gamma, \Delta \Rightarrow \Box A} \qquad \mathsf{k}_\Diamond\ \frac{\Gamma, B \Rightarrow A}{\Box\Gamma, \Delta, \Diamond B \Rightarrow \Diamond A}$$

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Soundness and completeness: $\vdash_{\mathsf{CK}} A$ iff $\vdash_{\mathsf{LCK}} \Rightarrow A$.

Main theorem

Realisation: If $\vdash_{\mathsf{LCK}} A'_1, \ldots, A'_n \Rightarrow C'$, a modal sequent, then there is a normal realisation $A_1, \ldots A_n \Rightarrow C$ of $A'_1, \ldots, A'_n \Rightarrow C'$ such that $\vdash_{\mathsf{JCK}} (A_1 \wedge \ldots \wedge A_n) \supset C$.

- if $t : A / \mu : A$ is a negative subformula of $A_1, \ldots A_n \Rightarrow C$, then t/μ is a proof/witness variable, and all these variables are pairwise distinct.

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The proof goes along the lines of that for the $\Box$-only fragment.

The operation $\sqcup$ on witness terms plays the same role as the operation $+$ on proof terms, i.e. to handle contractions of modal formulas.

Extensions

$$
\begin{tikzcd}
\mathsf{CT} \arrow[r, dash] \arrow[d, dash] & \mathsf{CS4} \arrow[d, dash] & \\
\mathsf{CD} \arrow[r, dash] \arrow[d, dash] & \mathsf{CD4} \arrow[r, dash] \arrow[d, dash] & \mathsf{CD45} \arrow[d, dash] \\
\mathsf{CK} \arrow[r, dash] & \mathsf{CK4} \arrow[r, dash] & \mathsf{CK45}
\end{tikzcd}
$$

$$
\begin{array}{rl}
\mathsf{d}: & \Box A \supset \Diamond A \\
\mathsf{t}: & (A \supset \Diamond A) \wedge (\Box A \supset A) \\
4: & (\Diamond\Diamond A \supset \Diamond A) \wedge (\Box A \supset \Box\Box A) \\
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In particular, the $\Box$-version of 4 requires the proof checker operator !

$$\mathsf{j4}_{\Box} : t : A \supset\, !t : t : A$$

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but *a priori* no additional operation for the $\Diamond$-version of 4.

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We think that the method here could be further extended,
but we would need to prove cut-elimination for the other systems.

Conclusions

In a nutshell:
We introduced witness terms and defined an operator combining proof terms and witness terms to realise the constructive modal axiom k_2.

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Future:

1. Intuitionistic modal logic IK = constructive CK +

   $\mathsf{k}_3 : \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B) \quad \mathsf{k}_4 : (\Diamond A \supset \Box B) \supset \Box(A \supset B) \quad \mathsf{k}_5 : \Diamond\bot \supset \bot$

   No ordinary sequent calculi for such logics, but there are nested sequent calculi for logics without axiom d. (Straßburger)

   - adapt the realisation proof for classical nested sequents calculi. (Goetschi and Kuznets)

2. Investigate the semantics of the logics we proposed.

   - adapt modular models. (Fitting)

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We introduced witness terms and defined an operator combining proof terms and witness terms to realise the constructive modal axiom k_2.

Future:

1. Intuitionistic modal logic IK = constructive CK +

   $\mathsf{k}_3: \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)$ $\quad \mathsf{k}_4: (\Diamond A \supset \Box B) \supset \Box(A \supset B)$ $\quad \mathsf{k}_5: \Diamond\bot \supset \bot$

   No ordinary sequent calculi for such logics, but there are nested sequent calculi for logics without axiom d. (Straßburger)

   - adapt the realisation proof for classical nested sequents calculi. (Goetschi and Kuznets)

2. Investigate the semantics of the logics we proposed.

   - adapt modular models. (Fitting)

Thank you. Let's discuss!