

Justification logic for constructive modal logic

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The big picture

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Justification logic:

Gödel:

What is the classical provability semantics of intuitionistic logic?

Artemov:

Logic of Proofs gives an operational view of this S4 type of provability.

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Extensions: realisation of logics below and above S4

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Intuitionistic variants: Some investigations toward

- ▶ realisation theorems (Artemov/Steren and Bonelli),
- ▶ epistemic semantics (Marti and Studer),
- ▶ and arithmetical completeness (Artemov and Iemhoff),

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but where the modal language is restricted to the \Box modality.

However, **intuitionistically** \Diamond cannot simply be viewed as the dual of \Box .

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Intuitionistic modal logic?

The program: represent the operational side of the intuitionistic \diamond .

The focus: on **constructive versions** of modal logic.

Constructive modal logic

Formulas: $A ::= \perp \mid a \mid A \wedge A \mid A \vee A \mid A \supset A$

Logic CK: Intuitionistic Propositional Logic

Constructive modal logic

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Logic CK: Intuitionistic Propositional Logic

+ $k_1: \Box(A \supset B) \supset (\Box A \supset \Box B)$ + necessitation: $\frac{A}{\Box A}$
+ $k_2: \Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$

(Wijesekera/Bierman and de Paiva/Mendler and Scheele)

Justification logic

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In the constructive version, we also add **witness terms** into the language.

$$\Diamond A \rightsquigarrow \mu : A \rightsquigarrow \mu \text{ is a witness of } A$$

Justification logic

Modal formulas: $A ::= \perp \mid a \mid A \wedge A \mid A \vee A \mid A \supset A \mid \Box A$

Justification formulas: $A ::= \perp \mid a \mid A \wedge A \mid A \vee A \mid A \supset A \mid t:A$

Grammar of terms:

$t ::= c \mid x \mid (t \cdot t) \mid (t + t) \mid !t$

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x : proof variables

\cdot : application

$+$: sum

$!$: *proof checker*

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α : witness variables

\star : execution

\sqcup : disjoint witness union

Justification logic for constructive modal logic

Axiomatisation JCK:

taut: Complete finite set of axioms for **intuitionistic** propositional logic

jk $_{\square}$: $t : (A \supset B) \supset (s : A \supset t \cdot s : B)$

sum: $s : A \supset (s + t) : A$ and $t : A \supset (s + t) : A$

$$\text{mp} \frac{A \supset B \quad A}{B}$$

$$\text{ian} \frac{A \text{ is an axiom instance}}{c_1 : \dots c_n : A}$$

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The machinery

Application: $jk_{\square} : t : (A \supset B) \supset (s : A \supset t \cdot s : B)$

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Witness execution: $jk_{\diamond} : t : (A \supset B) \supset (\mu : A \supset t \star \mu : B)$

If t is a proof of $A \supset B$ and μ is a witness for A , then the same model denoted $t \star \mu$ is also a witness for B .

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Sum and union: $s : A \supset (s + t) : A, \mu : A \supset (\mu \sqcup \nu) : B, \dots$

We adopt Artemov's $+$ to incorporate monotonicity of reasoning, and also transpose it on the witness side with \sqcup .

Iterated axiom necessitation and modus ponens:

The machinery

Justification logic can internalise its own reasoning.

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Lifting Lemma:

- ▶ If $A_1, \dots, A_n \vdash_{\text{JCK}} B$, then there exists a **proof term** $t(x_1, \dots, x_n)$ such that, for all terms s_1, \dots, s_n

$$\vdash_{\text{JCK}} s_1 : A_1 \wedge \dots \wedge s_n : A_n \supset t(s_1, \dots, s_n) : B$$

- ▶ If $A_1, \dots, A_n, C \vdash_{\text{JCK}} B$, then there exists a **witness term** $\mu(x_1, \dots, x_n, \beta)$ such that, for all terms s_1, \dots, s_n and ν

$$\vdash_{\text{JCK}} s_1 : A_1 \wedge \dots \wedge s_n : A_n \wedge \nu : C \supset \mu(s_1, \dots, s_n, \nu) : B$$

Correspondence

Forgetful projection: If $\vdash_{\text{JCK}} F$, then $\vdash_{\text{CK}} F^\circ$,

where $(\cdot)^\circ$ maps **justification formulas onto modal formulas**, in particular:

$$(t : A)^\circ := \Box A^\circ$$

$$(\mu : A)^\circ := \Diamond A^\circ$$

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Standard solution:

Consider a proof of the modal theorem in a **cut-free sequent calculus**.

Sequent calculus for modal logic

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Sequent system LCK:

$$\text{id} \frac{}{\Gamma, a \Rightarrow a}$$

$$\perp_L \frac{}{\Gamma, \perp \Rightarrow C}$$

$$\vee_L \frac{\Gamma, A \Rightarrow C \quad \Gamma, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C}$$

$$\vee_R \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B} \quad \vee_R \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}$$

$$\wedge_L \frac{\Gamma, A, B \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C}$$

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$$\supset_L \frac{\Gamma, A \supset B \Rightarrow A \quad \Gamma, B \Rightarrow C}{\Gamma, A \supset B \Rightarrow C}$$

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$$k_{\Box} \frac{\Gamma \Rightarrow A}{\Box \Gamma, \Delta \Rightarrow \Box A}$$

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Soundness and completeness: $\vdash_{CK} A$ iff $\vdash_{LCK} \Rightarrow A$.

Main theorem

Realisation: If $\vdash_{\text{LCK}} A'_1, \dots, A'_n \Rightarrow C'$, a modal sequent, then there is a **normal realisation** $A_1, \dots, A_n \Rightarrow C$ of $A'_1, \dots, A'_n \Rightarrow C'$ such that $\vdash_{\text{JCK}} (A_1 \wedge \dots \wedge A_n) \supset C$.

- ▶ if $t : A/\mu : A$ is a negative subformula of $A_1, \dots, A_n \Rightarrow C$, then t/μ is a proof/witness variable, and all these variables are pairwise distinct.

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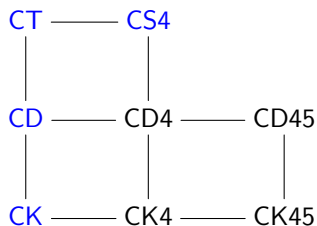
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The proof goes along the lines of that for the \Box -only fragment.

The operation \sqcup on witness terms plays the same role as the operation $+$ on proof terms, i.e. to handle contractions of modal formulas.

Extensions



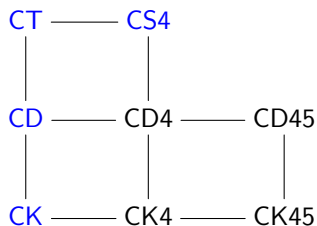
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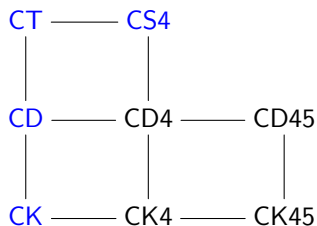
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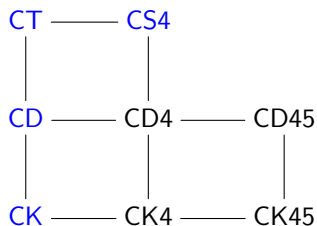
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In particular, the \Box -version of 4 requires the **proof checker** operator !

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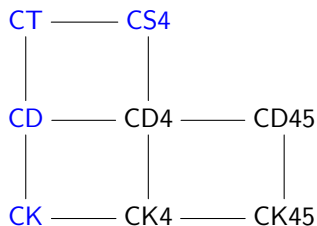
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We think that the method here could be further extended,
but we would need to prove **cut-elimination** for the other systems.

Conclusions

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We introduced **witness terms** and defined an operator **combining** proof terms and witness terms to realise the **constructive** modal axiom k_2 .

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Future:

1. **Intuitionistic** modal logic $IK = \text{constructive CK} +$

$$k_3: \diamond(A \vee B) \supset (\diamond A \vee \diamond B) \quad k_4: (\diamond A \supset \Box B) \supset \Box(A \supset B) \quad k_5: \diamond \perp \supset \perp$$

No ordinary sequent calculi for such logics, but there are **nested sequent calculi** for logics without axiom d. (Straßburger)

- ▶ adapt the realisation proof for classical nested sequents calculi. (Goetschi and Kuznets)
2. Investigate the **semantics** of the logics we proposed.
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Thank you. Let's discuss!

