Justification logic for constructive modal logic

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The big picture

Justification logic:

Gödel: What is the classical provability semantics of intuitionistic logic?

Artemov: Logic of Proofs gives an operational view of this S4 type of provability.

Semantics: Peano arithmetics or epistemic possible worlds models

Extensions: realisation of logics below and above S4

Intuitionistic variants:

▶ realisation theorems (Artemov/Steren and Bonelli),
▶ epistemic semantics (Marti and Studer),
▶ and arithmetical completeness (Artemov and Iemhoff),

However, intuitionistically cannot simply be viewed as the dual of modality.
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▶ epistemic semantics (Marti and Studer),
▶ and arithmetical completeness (Artemov and Iemhoff),
but where the modal language is restricted to the \( \Box \) modality.

However, intuitionistically \( \Diamond \) cannot simply be viewed as the dual of \( \Box \).
What are we doing here?

Justifying ◊:
We start with Artemov’s treatment of the □-fragment of intuitionistic modal logic.
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We start with Artemov’s treatment of the $\square$-fragment of intuitionistic modal logic.

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Intuitionistic modal logic?
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Intuitionistic modal logic?
The program: represent the operational side of the intuitionistic ◊.
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We start with Artemov’s treatment of the □-fragment of intuitionistic modal logic.

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Intuitionistic modal logic?
The program: represent the operational side of the intuitionistic ◊.

The focus: on constructive versions of modal logic.
Constructive modal logic

**Formulas:** \( A ::= \bot \mid a \mid A \land A \mid A \lor A \mid A \supset A \)

**Logic CK:** Intuitionistic Propositional Logic
Constructive modal logic

Formulas: \( A ::= \bot \mid a \mid A \land A \mid A \lor A \mid A \supset A \mid \Box A \mid \Diamond A \)

Logic CK: Intuitionistic Propositional Logic

\[ \begin{align*}
+ & \quad k_1: \quad \Box(A \supset B) \supset (\Box A \supset \Box B) \\
+ & \quad k_2: \quad \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) \\
+ & \quad \text{necessitation:} \quad \frac{A}{\Box A}
\end{align*} \]

(Wijesekera/Bierman and de Paiva/Mendler and Scheele)
Justification logic adds **proof terms** directly inside its language.

\[
\square A \quad \leadsto \quad t : A \quad \leadsto \quad t \text{ is a proof of } A
\]
Justification logic adds proof terms directly inside its language.

\( \square A \; \Rightarrow \; t : A \; \Rightarrow \; t \text{ is a proof of } A \)

In the constructive version, we also add witness terms into the language.

\( \Diamond A \; \Rightarrow \; \mu : A \; \Rightarrow \; \mu \text{ is a witness of } A \)
Justification logic

Modal formulas: \[ A ::= \bot \mid a \mid A \land A \mid A \lor A \mid A \supset A \mid \Box A \]

Justification formulas: \[ A ::= \bot \mid a \mid A \land A \mid A \lor A \mid A \supset A \mid t : A \]

Grammar of terms:

\[
t ::= c \mid x \mid (t \cdot t) \mid (t + t) \mid !t
\]

- \(c\): proof constants
- \(x\): proof variables
- \(\cdot\): application
- \(+\): sum
- \(!\): proof checker
Justification logic for constructive modal logic

Modal formulas: $A ::= \bot \mid a \mid A \land A \mid A \lor A \mid A \supset A \mid \Box A \mid \Diamond A$

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\[ t ::= c \mid x \mid (t \cdot t) \mid (t + t) \mid !t \]
\[ \mu ::= \alpha \mid t \ast \mu \mid (\mu \sqcup \mu) \]

\( c \): proof constants
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Justification logic for constructive modal logic

Modal formulas: \[ A ::= \bot | a | A \land A | A \lor A | A \supset A | \Box A | \Diamond A \]

Justification formulas: \[ A ::= \bot | a | A \land A | A \lor A | A \supset A | t : A | \mu : A \]

Grammar of terms:

\[
\begin{align*}
t ::= & \ c \ | \ x \ | \ (t \cdot t) \ | \ (t + t) \ | \ ! t \\
\mu ::= & \ \alpha \ | \ t \star \mu \ | \ (\mu \sqcup \mu)
\end{align*}
\]

c : proof constants
x : proof variables
\cdot : application
+ : sum
! : proof checker

\alpha : witness variables
\star : execution
\sqcup : disjoint witness union
Justification logic for constructive modal logic

**Axiomatisation JCK:**

- **taut:** Complete finite set of axioms for intuitionistic propositional logic
- **jk□:** \( t : (A ⊃ B) ⊃ (s : A ⊃ t \cdot s : B) \)
- **sum:** \( s : A ⊃ (s + t) : A \) and \( t : A ⊃ (s + t) : A \)

\[
\text{mp} \quad \frac{A ⊃ B \quad A}{B} \quad \quad \text{ian} \quad \frac{A \text{ is an axiom instance}}{c_1 : \ldots c_n : A}
\]
Justification logic for constructive modal logic

**Axiomatisation** JCK:

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\( k\): \( t : (A \supset B) \supset (\mu : A \supset t \cdot \mu : B) \)
- **sum**: \( s : A \supset (s + t) : A \) and \( t : A \supset (s + t) : A \)
- **union**: \( \mu : A \supset (\mu \sqcup \nu) : A \) and \( \nu : A \supset (\mu \sqcup \nu) : A \)

\[ \text{mp} \quad \frac{A \supset B \quad A}{B} \quad \text{ian} \quad \frac{A \text{ is an axiom instance}}{c_1 : \ldots : c_n : A} \]
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- **taut**: Complete finite set of axioms for intuitionistic propositional logic
  
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The machinery

Application:  \( jk_\square : t : (A \supset B) \supset (s : A \supset t \cdot s : B) \)

If \( t \) is a proof of \( A \supset B \) and \( s \) is a proof of \( A \), then \( t \cdot s \) is a proof of \( B \).
The machinery

**Application:** \[ jk : t : (A \supset B) \supset (s : A \supset t \cdot s : B) \]

If \( t \) is a proof of \( A \supset B \) and \( s \) is a proof of \( A \), then \( t \cdot s \) is a proof of \( B \).

**Witness execution:** \[ jk : t : (A \supset B) \supset (\mu : A \supset t \circ \mu : B) \]

If \( t \) is a proof of \( A \supset B \) and \( \mu \) is a witness for \( A \), then the same model denoted \( t \circ \mu \) is also a witness for \( B \).
**The machinery**

**Application:** \( jk_t : t : (A \supset B) \supset (s : A \supset t \cdot s : B) \)

If \( t \) is a proof of \( A \supset B \) and \( s \) is a proof of \( A \), then \( t \cdot s \) is a proof of \( B \).

**Witness execution:** \( jk_{\oplus} : t : (A \supset B) \supset (\mu : A \supset t \ast \mu : B) \)

If \( t \) is a proof of \( A \supset B \) and \( \mu \) is a witness for \( A \), then the same model denoted \( t \ast \mu \) is also a witness for \( B \).

**Sum and union:** \( s : A \supset (s + t) : A, \mu : A \supset (\mu \sqcup \nu) : B, \ldots \)

We adopt Artemov’s \( + \) to incorporate monotonicity of reasoning, and also transpose it on the witness side with \( \sqcup \).
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We adopt Artemov’s \( + \) to incorporate monotonicity of reasoning, and also transpose it on the witness side with \( \sqcup \).

**Iterated axiom necessitation and modus ponens:**
Justification logic can internalise its own reasoning.
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**Lifting Lemma:**

- If $A_1, \ldots, A_n \vdash_{JCK} B$, then there exists a proof term $t(x_1, \ldots, x_n)$ such that, for all terms $s_1, \ldots, s_n$
  
  $$\vdash_{JCK} s_1 : A_1 \land \ldots \land s_n : A_n \supset t(s_1, \ldots, s_n) : B$$

- If $A_1, \ldots, A_n, C \vdash_{JCK} B$, then there exists a witness term $\mu(x_1, \ldots, x_n, \beta)$ such that, for all terms $s_1, \ldots, s_n$ and $\nu$
  
  $$\vdash_{JCK} s_1 : A_1 \land \ldots \land s_n : A_n \land \nu : C \supset \mu(s_1, \ldots, s_n, \nu) : B$$
Forgetful projection: If $\vdash_{JCK} F$, then $\vdash_{CK} F^\circ$, 

where $(\cdot)^\circ$ maps justification formulas onto modal formulas, in particular:

$$(t : A)^\circ := \Box A^\circ \quad (\mu : A)^\circ := \Diamond A^\circ$$
Forgetful projection: If $\vdash_{\text{JCK}} F$, then $\vdash_{\text{CK}} F^\circ$,

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Can we get the converse?
I.e. can every modal logic theorem be realised by a justification theorem.
Correspondence

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Idea:
Transform directly a Hilbert proof of a modal theorem into a Hilbert proof of its realisation in justification logic.
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**Idea:**
Transform directly a Hilbert proof of a modal theorem into a Hilbert proof of its realisation in justification logic.

**Problem:**
*Modus ponens* can create dependencies between modalities.

**Standard solution:**
Consider a proof of the modal theorem in a cut-free sequent calculus.
Sequent calculus for modal logic

\[ A_1, \ldots, A_n \Rightarrow C; (A_1 \land \ldots \land A_n) \supset C \]

\[ \Gamma, a \Rightarrow a \perp \]

\[ \Gamma, \perp \Rightarrow C \]

\[ \Gamma, A \Rightarrow C \]

\[ \Gamma, B \Rightarrow C \]

\[ \Gamma \Rightarrow A \lor R \]

\[ \Gamma \Rightarrow A \lor B \]

\[ \Gamma \Rightarrow A \land R \]

\[ \Gamma \Rightarrow A \land B \]

\[ \Gamma, A \supset B \Rightarrow C \]

\[ \Gamma, A \Rightarrow B \supset R \]

\[ \Gamma, A \Rightarrow B \supset R \]

\[ \Gamma, A \Rightarrow B \supset R \]

\[ \Gamma, A \Rightarrow B \supset R \]

\[ \Gamma \Rightarrow A_k \]

\[ \Gamma, \Delta \Rightarrow A_k \]

Soundness and completeness:

\[ \vdash_{CK} A \iff \vdash_{LCK} C \]
Sequent calculus for modal logic

Sequent system LCK:

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<th>Rule</th>
<th>Premises</th>
<th>Conclusion</th>
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<tr>
<td>id</td>
<td>$\Gamma, a \Rightarrow a$</td>
<td>$\Gamma, a \Rightarrow a$</td>
</tr>
<tr>
<td>$\perp_{L}$</td>
<td>$\Gamma, \perp \Rightarrow C$</td>
<td>$\Gamma, \perp \Rightarrow C$</td>
</tr>
<tr>
<td>$\lor_{L}$</td>
<td>$\Gamma, A \Rightarrow C, \Gamma, B \Rightarrow C$</td>
<td>$\Gamma, A \lor B \Rightarrow C$</td>
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<tr>
<td>$\lor_{R}$</td>
<td>$\Gamma \Rightarrow A \lor B$</td>
<td>$\Gamma \Rightarrow A \lor B$</td>
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<tr>
<td>$\land_{L}$</td>
<td>$\Gamma, A, B \Rightarrow C$</td>
<td>$\Gamma, A \land B \Rightarrow C$</td>
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</tr>
<tr>
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Soundness and completeness: $\vdash_{CK} A \iff \vdash_{LCK} \Rightarrow A$. 
Sequent calculus for modal logic

Sequent system \( \text{LCK} \):

\[ A_1, \ldots, A_n \Rightarrow C \sim (A_1 \land \ldots \land A_n) \supset C \]

- **id** \[ \Gamma, a \Rightarrow a \]
- **⊥L** \[ \Gamma, \bot \Rightarrow C \]
- **γL** \[ \Gamma, A \Rightarrow C \quad \Gamma, B \Rightarrow C \frac{}{\Gamma, A \lor B \Rightarrow C} \]
- **γR** \[ \Gamma \Rightarrow A \quad \Gamma \Rightarrow B \frac{}{\Gamma \Rightarrow A \lor B} \]
- **∧L** \[ \Gamma, A, B \Rightarrow C \frac{}{\Gamma, A \land B \Rightarrow C} \]
- **∧R** \[ \Gamma \Rightarrow A \quad \Gamma \Rightarrow B \frac{}{\Gamma \Rightarrow A \land B} \]
- **⊃L** \[ \Gamma, A \supset B \Rightarrow A \quad \Gamma, B \Rightarrow C \frac{}{\Gamma, A \supset B \Rightarrow C} \]
- **⊃R** \[ \Gamma, A \Rightarrow B \frac{}{\Gamma \Rightarrow A \supset B} \]
Sequent calculus for modal logic

Sequent system LCK:

\[
\begin{align*}
\text{id} & \quad \Gamma, a \Rightarrow a \\
\text{\textbackslash{}L} & \quad \Gamma, A \Rightarrow C, \Gamma, B \Rightarrow C \quad \Gamma, A \vee B \Rightarrow C \\
\text{\textbackslash{}R} & \quad \Gamma \Rightarrow A \quad \Gamma \Rightarrow B \\
\text{\textbackslash{}R} & \quad \Gamma \Rightarrow A \vee B \\
\text{\textgreek{a}L} & \quad \Gamma, A, B \Rightarrow C \\
\text{\textgreek{a}R} & \quad \Gamma \Rightarrow A \quad \Gamma \Rightarrow B \\
\text{\textgreek{c}L} & \quad \Gamma, A \dashv B \Rightarrow A \\
\text{\textgreek{c}R} & \quad \Gamma, A \Rightarrow B \\
\text{k\lozenge} & \quad \Gamma \Rightarrow A \\
\text{k\lozenge} & \quad \Gamma, B \Rightarrow A \\
\end{align*}
\]

Soundness and completeness: \( \vdash \text{CK} A \iff \vdash \text{LCK} A \Rightarrow A \).
Sequent calculus for modal logic

Sequent system \( \text{LCK} \):

\[
\begin{align*}
\text{id} & : \Gamma, a \Rightarrow a \\
\text{⊥}_L & : \Gamma, \bot \Rightarrow C \\
\land_L & : \Gamma, A, B \Rightarrow C \quad \land_L \quad \Gamma, A \land B \Rightarrow C \\
\lor_L & : \Gamma, A \Rightarrow C \quad \lor_L \quad \Gamma, B \Rightarrow C \\
\lor_R & : \Gamma \Rightarrow A \lor B \\
\land_R & : \Gamma \Rightarrow A \quad \land_R \quad \Gamma \Rightarrow A \land B \\
\supset_L & : \Gamma, A \supset B \Rightarrow A \quad \supset_L \quad \Gamma, B \Rightarrow C \\
\supset_R & : \Gamma, A \Rightarrow B \\
\kappa_\Box & : \Gamma \Rightarrow A \quad \kappa_\Box \quad \Box \Gamma, A \Rightarrow \Box A \\
\kappa_\Diamond & : \Gamma, B \Rightarrow A \quad \kappa_\Diamond \quad \Box \Gamma, \Box B \Rightarrow \Box A \\
\end{align*}
\]

Soundness and completeness: \( \vdash_{\text{CK}} A \) iff \( \vdash_{\text{LCK}} \Rightarrow A \).
Main theorem

**Realisation:** If \( \vdash_{\text{LCK}} A'_1, \ldots, A'_n \Rightarrow C' \), a modal sequent, then there is a normal realisation \( A_1, \ldots A_n \Rightarrow C \) of \( A'_1, \ldots, A'_n \Rightarrow C' \) such that \( \vdash_{\text{JCK}} (A_1 \land \ldots \land A_n) \supset C \).

- if \( t : A/\mu : A \) is a negative subformula of \( A_1, \ldots A_n \Rightarrow C \), then \( t/\mu \) is a proof/witness variable, and all these variables are pairwise distinct.
Main theorem

**Realisation:** If $\vdash_{\text{LCK}} A'_1, \ldots, A'_n \Rightarrow C'$, a modal sequent, then there is a **normal realisation** $A_1, \ldots A_n \Rightarrow C$ of $A'_1, \ldots, A'_n \Rightarrow C'$ such that $\vdash_{\text{JCK}} (A_1 \land \ldots \land A_n) \supset C$.

- if $t : A/\mu : A$ is a negative subformula of $A_1, \ldots A_n \Rightarrow C$, then $t/\mu$ is a proof/witness variable, and all these variables are pairwise distinct.

The proof goes along the lines of that for the $\Box$-only fragment.

The operation $\sqcup$ on witness terms plays the same role as the operation $+$ on proof terms, i.e. to handle contractions of modal formulas.
Extensions

CT — CS4

CD — CD4 — CD45

CK — CK4 — CK45

d: $\Box A \supset \Diamond A$

t: $(A \supset \Diamond A) \land (\Box A \supset A)$

4: $(\Diamond \Diamond A \supset \Diamond A) \land (\Box A \supset \Box \Box A)$

5: $(\Diamond A \supset \Box \Diamond A) \land (\Diamond \Box A \supset \Box A)$

No other operation on witness terms outside execution and disjoint union. In particular, the $2$-version of 4 requires the proof checker operator $!_{2}$.

but a priori no additional operation for the $3$-version of 4.

We think that the method here could be further extended, but we would need to prove cut-elimination for the other systems.
No other operation on witness terms outside execution and disjoint union.
Extensions

CT ——— CS4
|
CD ——— CD4 ——— CD45
|
CK ——— CK4 ——— CK45

d: \( \Box A \supset \Diamond A \)
t: \((A \supset \Diamond A) \land (\Box A \supset A)\)
4: \((\Diamond \Diamond A \supset \Diamond A) \land (\Box A \supset \Box A)\)
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No other operation on witness terms outside execution and disjoint union. In particular, the \(\Box\)-version of 4 requires the proof checker operator \(!\)

\[ j_{A \Box} : t : A \supset ! t : t : A \]
Extensions

No other operation on witness terms outside execution and disjoint union. In particular, the $\square$-version of 4 requires the proof checker operator $!$

$$j_{4\square}: t: A \supset !t: t: A$$

but a priori no additional operation for the $\Diamond$-version of 4.

$$j_{4\Diamond}: \mu: \nu: A \supset \nu: A$$
No other operation on witness terms outside execution and disjoint union. In particular, the $\square$-version of 4 requires the proof checker operator $!$

$$ j4_{\square} : t : A \supset ! t : t : A $$

but a priori no additional operation for the $\Diamond$-version of 4.

$$ j4_{\Diamond} : \mu : \nu : A \supset \nu : A $$

We think that the method here could be further extended, but we would need to prove cut-elimination for the other systems.
Conclusions

In a nutshell:
We introduced witness terms and defined an operator combining proof terms and witness terms to realise the constructive modal axiom $k_2$.

Future:
1. Intuitionistic modal logic $IK = \text{constructive } CK + \text{ } k_3$
   \[ (A \lor B) \supset (3A \lor 3B) \]
   \[ (3A \supset 2B) \supset 2(A \supset B) \]
   \[ 3\bot \supset \bot \]
2. Investigate the semantics of the logics we proposed.

Thank you. Let's discuss!
Conclusions

In a nutshell:
We introduced witness terms and defined an operator combining proof terms and witness terms to realise the constructive modal axiom $k_2$.

Future:
1. Intuitionistic modal logic IK = constructive CK +
   
   $k_3 : \Diamond (A \lor B) \supset (\Diamond A \lor \Diamond B)$    
   $k_4 : (\Diamond A \supset \Box B) \supset \Box (A \supset B)$    
   $k_5 : \Diamond \bot \supset \bot$

   No ordinary sequent calculi for such logics, but there are nested sequent calculi for logics without axiom d. (Straßburger)
   
   ▶ adapt the realisation proof for classical nested sequents calculi. (Goetschi and Kuznets)

2. Investigate the semantics of the logics we proposed.
   
   ▶ adapt modular models. (Fitting)
Conclusions

In a nutshell:
We introduced witness terms and defined an operator combining proof terms and witness terms to realise the constructive modal axiom $k_2$.

Future:

1. Intuitionistic modal logic $IK = \text{constructive } CK +$
   - $k_3 : \Diamond (A \lor B) \supset (\Diamond A \lor \Diamond B)$
   - $k_4 : (\Diamond A \lor \Box B) \supset \Box (A \lor B)$
   - $k_5 : \Diamond \bot \supset \bot$

   No ordinary sequent calculi for such logics, but there are nested sequent calculi for logics without axiom d. (Straßburger)
   - adapt the realisation proof for classical nested sequents calculi. (Goetschi and Kuznets)

2. Investigate the semantics of the logics we proposed.
   - adapt modular models. (Fitting)

Thank you. Let’s discuss!