Focused Nested Sequents

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Focusing for nested sequents

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→ Focusing :
distillation process for proofs in linear, intuitionistic and classical logic

Focusing for nested sequents

$\rightarrow$ Focusing :
distillation process for proofs in linear, intuitionistic and classical logic

$\rightarrow$ Nested sequents : modular proof systems for modal logics

Classical modal logic

Formulas: $A ::= a \mid \bar{a} \mid A \wedge A \mid A \vee A \mid \Box A \mid \Diamond A$

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Axioms for K**:** classical propositional logic and

$$k\colon \Box(A \to B) \to (\Box A \to \Box B)$$

Rules: modus ponens: $\dfrac{A \quad A \to B}{B}$ necessitation: $\dfrac{A}{\Box A}$

Classical modal logic

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$$k\colon \Box(A \to B) \to (\Box A \to \Box B)$$

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The S5**-cube:**

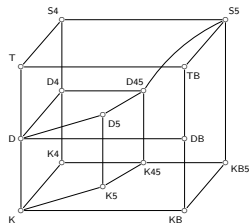d: $\Box A \to \Diamond A$
t: $A \to \Diamond A$
b: $A \to \Box \Diamond A$
4: $\Diamond \Diamond A \to \Diamond A$
5: $\Diamond A \to \Box \Diamond A$

Nested sequents

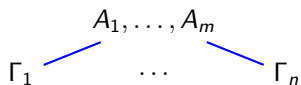**Sequent:** $\Gamma ::= A_1, \ldots, A_m$

$$fm(\Gamma) = A_1 \vee \ldots \vee A_m$$

Nested sequents

Nested sequent: $\Gamma ::= A_1, \ldots, A_m, [\Gamma_1], \ldots, [\Gamma_n]$

$$fm(\Gamma) = A_1 \vee \ldots \vee A_m \vee \Box fm(\Gamma_1) \vee \ldots \vee \Box fm(\Gamma_n)$$

$$
\begin{array}{ccc}
& A_1, \ldots, A_m & \\
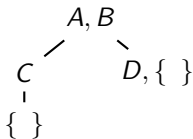\Gamma_1 & \cdots & \Gamma_n
\end{array}
$$

[Kashima, 1994], [Brünnler, 2009], [Poggiolesi, 2009]

Nested sequents

Nested sequent:
$$\Gamma ::= A_1, \ldots, A_m, [\Gamma_1], \ldots, [\Gamma_n]$$
$$fm(\Gamma) = A_1 \vee \ldots \vee A_m \vee \Box fm(\Gamma_1) \vee \ldots \vee \Box fm(\Gamma_n)$$

Sequent context: $\Gamma\{\ \}\{\ \} = A, B, [C, [\{\ \}]], [D, \{\ \}]$

[Kashima, 1994], [Brünnler, 2009], [Poggiolesi, 2009]

Nested sequents

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Sequent context: $\Gamma\{B\}\{\ \} = A, B, [C, [B]], [D, \{\ \}]$

$$
\begin{array}{c}
A, B \\
\diagup \quad \diagdown \\
C \qquad \quad D, \{\ \} \\
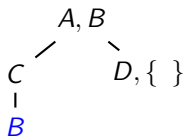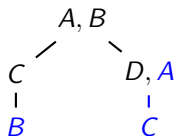| \\
B
\end{array}
$$

[Kashima, 1994], [Brünnler, 2009], [Poggiolesi, 2009]

Nested sequents

Nested sequent: $\Gamma ::= A_1, \ldots, A_m, [\Gamma_1], \ldots, [\Gamma_n]$

$fm(\Gamma) = A_1 \vee \ldots \vee A_m \vee \Box fm(\Gamma_1) \vee \ldots \vee \Box fm(\Gamma_n)$

Sequent context: $\Gamma\{B\}\{A, [C]\} = A, B, [C, [B]], [D, A, [C]]$

[Kashima, 1994], [Brünnler, 2009], [Poggiolesi, 2009]

The standard nested system

Formulas: $A \quad ::= \quad a \mid \bar{a} \mid A \wedge A \mid A \vee A \mid \Box A \mid \Diamond A$

System KN:

$$\text{cont } \frac{\Gamma\{A, A\}}{\Gamma\{A\}} \qquad \Box \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \vee \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}}$$

$$\text{id } \frac{}{\Gamma\{a, \bar{a}\}} \qquad \text{k}^\Diamond \frac{\Gamma\{[A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} \quad \wedge \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}}$$

$$\text{k}: \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$$

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$$\text{k}: \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$$

Modal rules:

$$\text{d}^\Diamond \frac{\Gamma\{[A]\}}{\Gamma\{\Diamond A\}} \qquad \text{t}^\Diamond \frac{\Gamma\{A\}}{\Gamma\{\Diamond A\}} \qquad \text{b}^\Diamond \frac{\Gamma\{[\Delta], A\}}{\Gamma\{[\Delta, \Diamond A]\}} \qquad 4^\Diamond \frac{\Gamma\{[\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} \qquad 5^\Diamond \frac{\Gamma\{\emptyset\}\{\Diamond A\}}{\Gamma\{\Diamond A\}\{\emptyset\}}$$

$$\text{d}: \Box A \rightarrow \Diamond A \qquad \text{t}: A \rightarrow \Diamond A \qquad \text{b}: A \rightarrow \Box \Diamond A \qquad 4: \Diamond\Diamond A \rightarrow \Diamond A \qquad 5: \Diamond A \rightarrow \Box\Diamond A$$

[Brünnler, 2009]

Focusing and polarities

$$
\dfrac{
  \dfrac{
    \mathsf{k}^\diamond \dfrac{
      \mathsf{k}^\diamond \dfrac{
        \vee \dfrac{
          \wedge \dfrac{
            \mathsf{id} \ \overline{[a, \bar{a}, \bar{b}, c]} \quad \mathsf{id} \ \overline{[b, \bar{a}, \bar{b}, c]}
          }{[a, \bar{a} \wedge b, \bar{b}, c]}
        }{[a, \bar{a} \wedge b, \bar{b} \vee c]}
      }{\diamond a, [\bar{a} \wedge b, \bar{b} \vee c]}
    }{\diamond a, \diamond(\bar{a} \wedge b), [\bar{b} \vee c]}
  }{}
}{}
$$

Focusing and polarities

Polarities:
	invertible rules	:	negative connectives
	non-invertible rules	:	Positive connectives

[Andreoli, 1992], [Laurent, 2004]

Focusing and polarities

Weak focusing: For any subproof $\overset{\pi'}{\underset{\Gamma\{P\}}{\parallel}}$ the only positive rules between two rules decomposing P are rules decomposing P.

$$
\mathsf{k}^\diamond \cfrac{
\mathsf{k}^\diamond \cfrac{
\wedge \cfrac{
\mathsf{k}^\diamond \cfrac{
\mathsf{id}\ \cfrac{}{[a, \bar{a}, \bar{b}, c]}
}{\diamond a, [\bar{a}, \bar{b}, c]}
\qquad
\mathsf{k}^\diamond \cfrac{
\mathsf{id}\ \cfrac{}{[a, b, \bar{b}, c]}
}{\diamond a, [b, \bar{b}, c]}
}{\diamond a, [\bar{a} \wedge b, \bar{b}, c]}
}{\vee\ \diamond a, [\bar{a} \wedge b, \bar{b} \vee c]}
}{\diamond a, \diamond(\bar{a} \wedge b), [\bar{b} \vee c]}
$$

[Andreoli, 1992], [Laurent, 2004]

Strong focusing: For any subproof $\pi' \|\atop \Gamma\{P\}$ the only rules between two rules decomposing P are rules decomposing P.

$$\cfrac{\cfrac{\cfrac{\cfrac{}{[a, \bar{a}, \bar{b}, c]} \text{ id}}{\Diamond a, [\bar{a}, \bar{b}, c]} \mathsf{k}^\Diamond}{\Diamond a, [\bar{a}, \bar{b} \vee c]} \vee \quad \cfrac{\cfrac{\cfrac{}{[a, b, \bar{b}, c]} \text{ id}}{\Diamond a, [b, \bar{b}, c]} \mathsf{k}^\Diamond}{\Diamond a, [b, \bar{b} \vee c]} \vee}{\cfrac{\Diamond a, [\bar{a} \wedge b, \bar{b} \vee c]}{\Diamond a, \Diamond(\bar{a} \wedge b), [\bar{b} \vee c]} \mathsf{k}^\Diamond} \wedge}$$

[Andreoli, 1992], [Laurent, 2004]

Focusing and polarities

Inversion: For any subproof
$$\pi' \Big\| \atop \Gamma\{N\}$$
the last rule is negative.

$$\wedge \frac{\text{k}^\diamond \frac{\text{id} \overline{[a, \bar{a}, \bar{b}, c]}}{\diamond a, [\bar{a}, \bar{b}, c]} \quad \text{k}^\diamond \frac{\text{id} \overline{[a, b, \bar{b}, c]}}{\diamond a, [b, \bar{b}, c]}}{\text{k}^\diamond \frac{\diamond a, [\bar{a} \wedge b, \bar{b}, c]}{\diamond a, \diamond(\bar{a} \wedge b), [\bar{b}, c]}} \\ \vee \frac{}{\diamond a, \diamond(\bar{a} \wedge b), [\bar{b} \vee c]}$$

[Andreoli, 1992], [Laurent, 2004]

The standard nested system

Formulas: $\quad A \quad ::= \quad a \mid \bar{a} \mid A \wedge A \mid A \vee A \mid \Box A \mid \Diamond A$

System KN:

$$\text{cont } \frac{\Gamma\{A, A\}}{\Gamma\{A\}} \qquad \Box \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \vee \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}}$$

$$\text{id } \frac{}{\Gamma\{a, \bar{a}\}} \qquad \mathsf{k}^\Diamond \frac{\Gamma\{[A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} \qquad \wedge \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}}$$

Modal rules:

$$\mathsf{d}^\Diamond \frac{\Gamma\{[A]\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{t}^\Diamond \frac{\Gamma\{A\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{b}^\Diamond \frac{\Gamma\{[\Delta], A\}}{\Gamma\{[\Delta, \Diamond A]\}} \qquad 4^\Diamond \frac{\Gamma\{[\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} \qquad 5^\Diamond \frac{\Gamma\{\emptyset\}\{\Diamond A\}}{\Gamma\{\Diamond A\}\{\emptyset\}}$$

The focused nested system

Formulas: A $::=$ $a \mid \bar{a} \mid A \wedge A \mid A \vee A \mid \Box A \mid \Diamond A$

System KN:

$$\text{cont} \, \frac{\Gamma\{A, A\}}{\Gamma\{A\}} \qquad \Box \, \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \vee \, \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}}$$

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The focused nested system

Polarized formulas:
$$P ::= a \mid \downarrow N \mid \Diamond P \mid P \wedge P$$
$$N ::= \bar{a} \mid \uparrow P \mid \Box N \mid N \vee N$$

System KN:

$$\text{cont } \frac{\Gamma\{A, A\}}{\Gamma\{A\}} \qquad \Box \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \vee \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}}$$

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Focused system KNF:

$$\text{dec} \; \frac{\Gamma\{P, \langle P \rangle\}}{\Gamma\{P\}} \qquad \Box \; \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \vee \; \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}}$$

$$\text{id} \; \frac{}{\Gamma\{\bar{a}, \langle a \rangle\}} \qquad \mathsf{k}^{\Diamond} \; \frac{\Gamma\{[\langle A \rangle, \Delta]\}}{\Gamma\{\langle \Diamond A \rangle, [\Delta]\}} \qquad \wedge \; \frac{\Gamma\{\langle A \rangle\} \quad \Gamma\{\langle B \rangle\}}{\Gamma\{\langle A \wedge B \rangle\}}$$

Modal rules:

$$\mathsf{d}^{\Diamond} \; \frac{\Gamma\{[A]\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{t}^{\Diamond} \; \frac{\Gamma\{A\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{b}^{\Diamond} \; \frac{\Gamma\{[\Delta], A\}}{\Gamma\{[\Delta, \Diamond A]\}} \qquad 4^{\Diamond} \; \frac{\Gamma\{[\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} \qquad 5^{\Diamond} \; \frac{\Gamma\{\emptyset\}\{\Diamond A\}}{\Gamma\{\Diamond A\}\{\emptyset\}}$$

The focused nested system

Polarized formulas:
$$P \quad ::= \quad a \mid {\downarrow}N \mid {\Diamond}P \mid P {\wedge} P$$
$$N \quad ::= \quad \bar{a} \mid {\uparrow}P \mid {\Box}N \mid N {\vee} N$$

Focused system KNF:

$$\text{dec} \; \frac{\Gamma\{P, \langle P \rangle\}}{\Gamma\{P\}} \qquad \Box \; \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \vee \; \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}} \qquad\qquad\qquad \text{sto} \; \frac{\Gamma\{P\}}{\Gamma\{{\uparrow}P\}}$$

$$\text{id} \; \frac{}{\Gamma\{\bar{a}, \langle a \rangle\}} \qquad \mathsf{k}^{\Diamond} \; \frac{\Gamma\{[\langle A \rangle, \Delta]\}}{\Gamma\{\langle {\Diamond}A \rangle, [\Delta]\}} \qquad \wedge \; \frac{\Gamma\{\langle A \rangle\} \quad \Gamma\{\langle B \rangle\}}{\Gamma\{\langle A \wedge B \rangle\}}$$

Modal rules:

$$\mathsf{d}^{\Diamond} \; \frac{\Gamma\{[A]\}}{\Gamma\{{\Diamond}A\}} \qquad \mathsf{t}^{\Diamond} \; \frac{\Gamma\{A\}}{\Gamma\{{\Diamond}A\}} \qquad \mathsf{b}^{\Diamond} \; \frac{\Gamma\{[\Delta], A\}}{\Gamma\{[\Delta, {\Diamond}A]\}} \qquad 4^{\Diamond} \; \frac{\Gamma\{[{\Diamond}A, \Delta]\}}{\Gamma\{{\Diamond}A, [\Delta]\}} \qquad 5^{\Diamond} \; \frac{\Gamma\{\emptyset\}\{{\Diamond}A\}}{\Gamma\{{\Diamond}A\}\{\emptyset\}}$$

The focused nested system

Polarized formulas:
$$P ::= a \mid \downarrow N \mid \Diamond P \mid P \wedge P$$
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Focused system KNF:

$$\text{dec } \frac{\Gamma\{P, \langle P \rangle\}}{\Gamma\{P\}} \qquad \Box \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \vee \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}} \qquad\qquad \text{sto } \frac{\Gamma\{P\}}{\Gamma\{\uparrow P\}}$$

$$\text{id } \frac{}{\Gamma\{\bar{a}, \langle a \rangle\}} \qquad \mathsf{k}^{\Diamond} \frac{\Gamma\{[\langle A \rangle, \Delta]\}}{\Gamma\{\langle \Diamond A \rangle, [\Delta]\}} \qquad \wedge \frac{\Gamma\{\langle A \rangle\} \quad \Gamma\{\langle B \rangle\}}{\Gamma\{\langle A \wedge B \rangle\}} \qquad\qquad \text{rel } \frac{\Gamma\{N\}}{\Gamma\{\langle \downarrow N \rangle\}}$$

Modal rules:

$$\mathsf{d}^{\Diamond} \frac{\Gamma\{[A]\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{t}^{\Diamond} \frac{\Gamma\{A\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{b}^{\Diamond} \frac{\Gamma\{[\Delta], A\}}{\Gamma\{[\Delta, \Diamond A]\}} \qquad \mathsf{4}^{\Diamond} \frac{\Gamma\{[\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} \qquad \mathsf{5}^{\Diamond} \frac{\Gamma\{\emptyset\}\{\Diamond A\}}{\Gamma\{\Diamond A\}\{\emptyset\}}$$

The focused nested system

Polarized formulas:

$$P ::= a \mid {\downarrow}N \mid \Diamond P \mid P \overset{\star}{\wedge} P \mid P \overset{\star}{\vee} P$$
$$N ::= \bar{a} \mid {\uparrow}P \mid \Box N \mid N \bar{\vee} N \mid N \bar{\wedge} N$$

Focused system KNF:

$$\text{dec} \frac{\Gamma\{P, \langle P \rangle\}}{\Gamma\{P\}} \qquad \Box \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \bar{\vee} \frac{\Gamma\{A, B\}}{\Gamma\{A \bar{\vee} B\}} \qquad \bar{\wedge} \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \bar{\wedge} B\}} \qquad \text{sto} \frac{\Gamma\{P\}}{\Gamma\{{\uparrow}P\}}$$

$$\text{id} \frac{}{\Gamma\{\bar{a}, \langle a \rangle\}} \qquad \mathsf{k}^\Diamond \frac{\Gamma\{[\langle A \rangle, \Delta]\}}{\Gamma\{\langle \Diamond A \rangle, [\Delta]\}} \qquad \overset{\star}{\wedge} \frac{\Gamma\{\langle A \rangle\} \quad \Gamma\{\langle B \rangle\}}{\Gamma\{\langle A \overset{\star}{\wedge} B \rangle\}} \qquad \overset{\star}{\vee}_i \frac{\Gamma\{\langle A_i \rangle\}}{\Gamma\{\langle A_1 \overset{\star}{\vee} A_2 \rangle\}} \qquad \text{rel} \frac{\Gamma\{N\}}{\Gamma\{\langle {\downarrow}N \rangle\}}$$

Modal rules:

$$\mathsf{d}^\Diamond \frac{\Gamma\{[A]\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{t}^\Diamond \frac{\Gamma\{A\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{b}^\Diamond \frac{\Gamma\{[\Delta], A\}}{\Gamma\{[\Delta, \Diamond A]\}} \qquad 4^\Diamond \frac{\Gamma\{[\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} \qquad 5^\Diamond \frac{\Gamma\{\emptyset\}\{\Diamond A\}}{\Gamma\{\Diamond A\}\{\emptyset\}}$$

The focused nested system

Polarized formulas:
$$P ::= a \mid {\downarrow}N \mid \Diamond P \mid P \overset{+}{\wedge} P \mid P \overset{+}{\vee} P$$
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Focused system KNF:

$$\text{dec} \frac{\Gamma\{P, \langle P \rangle\}}{\Gamma\{P\}} \qquad \Box \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \bar{\vee} \frac{\Gamma\{A, B\}}{\Gamma\{A \bar{\vee} B\}} \qquad \bar{\wedge} \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \bar{\wedge} B\}} \qquad \text{sto} \frac{\Gamma\{P\}}{\Gamma\{{\uparrow}P\}}$$

$$\text{id} \frac{}{\Gamma\{\bar{a}, \langle a \rangle\}} \qquad \text{k}^\Diamond \frac{\Gamma\{[\langle A \rangle, \Delta]\}}{\Gamma\{\langle \Diamond A \rangle, [\Delta]\}} \qquad \overset{+}{\wedge} \frac{\Gamma\{\langle A \rangle\} \quad \Gamma\{\langle B \rangle\}}{\Gamma\{\langle A \overset{+}{\wedge} B \rangle\}} \qquad \overset{+}{\vee}_i \frac{\Gamma\{\langle A_i \rangle\}}{\Gamma\{\langle A_1 \overset{+}{\vee} A_2 \rangle\}} \qquad \text{rel} \frac{\Gamma\{N\}}{\Gamma\{\langle {\downarrow}N \rangle\}}$$

Focused modal rules:

$$\text{d}^\Diamond \frac{\Gamma\{[\langle A \rangle]\}}{\Gamma\{\langle \Diamond A \rangle\}} \qquad \text{t}^\Diamond \frac{\Gamma\{\langle A \rangle\}}{\Gamma\{\langle \Diamond A \rangle\}} \qquad \text{b}^\Diamond \frac{\Gamma\{[\Delta], \langle A \rangle\}}{\Gamma\{[\Delta, \langle \Diamond A \rangle]\}} \qquad 4^\Diamond \frac{\Gamma\{[\langle \Diamond A \rangle, \Delta]\}}{\Gamma\{\langle \Diamond A \rangle, [\Delta]\}} \qquad 5^\Diamond \frac{\Gamma\{\emptyset\}\{\langle \Diamond A \rangle\}}{\Gamma\{\langle \Diamond A \rangle\}\{\emptyset\}}$$

Completeness of focusing

Let $X \subseteq \{d, t, b, 4, 5\}$.
If A is provable in $KN + X^\diamond$, then any $\mathrm{pol}(A)$ is provable in $KNF + X^\diamond$.

Completeness of focusing

Let $X \subseteq \{d, t, b, 4, 5\}$.
If A is provable in $\mathsf{KN} + X^\diamond$, then any $\mathrm{pol}(A)$ is provable in $\mathsf{KNF} + X^\diamond$.
Via cut-elimination:

$$\mathsf{cut} \ \frac{\Gamma\{P\} \quad \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}}$$

Completeness of focusing

Let $X \subseteq \{d, t, b, 4, 5\}$.
If A is provable in $\mathsf{KN} + X^\diamond$, then any $\mathrm{pol}(A)$ is provable in $\mathsf{KNF} + X^\diamond$.
Via cut-elimination:

$$\mathsf{cut}\ \frac{\Gamma\{P\} \quad \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}}$$

$$\mathsf{KN} \xrightarrow{\quad \text{simulation} \quad} \mathsf{KNF} + \mathsf{cut}$$

$$\mathsf{cut}\ \frac{\Gamma\{P\} \quad \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}}$$

Completeness of focusing

Let $X \subseteq \{d, t, b, 4, 5\}$.

If A is provable in $KN + X^\diamond$, then any $\mathrm{pol}(A)$ is provable in $KNF + X^\diamond$.

Via cut-elimination:

$$\mathrm{cut} \frac{\Gamma\{P\} \quad \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}}$$

$$KN \longrightarrow KNF + \mathrm{cut} \xrightarrow{\quad \text{cut-elimination} \quad} KNF$$

Completeness of focusing

Let $X \subseteq \{d, t, b, 4, 5\}$.
If A is provable in $KN + X^\diamond$, then any $\mathrm{pol}(A)$ is provable in KNF $+ X^\diamond$.
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Problem:

$$\text{cut} \frac{\Gamma\{P\} \quad \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}}$$

Completeness of focusing

Let $X \subseteq \{d, t, b, 4, 5\}$.

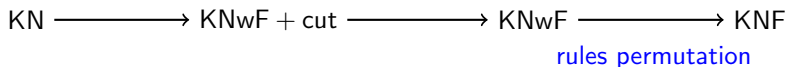If A is provable in $KN + X^\diamond$, then any $\mathrm{pol}(A)$ is provable in $KNF + X^\diamond$.

Via cut-elimination:

$$\mathrm{cut} \frac{\Gamma\{P\} \quad \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}}$$

Problem:

$\rightarrow$ **Weak focusing:**

$$KN \xrightarrow{\hspace{1cm}} KNwF + cut$$
$$\text{simulation}$$

$$\mathrm{cut} \frac{\Gamma\{P\} \quad \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}}$$

Completeness of focusing

Let $X \subseteq \{d, t, b, 4, 5\}$.
If A is provable in $KN + X^\diamond$, then any $\mathrm{pol}(A)$ is provable in $KNF + X^\diamond$.

Via cut-elimination:

$$\mathrm{cut} \frac{\Gamma\{P\} \quad \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}}$$

Problem:

$\rightarrow$ **Weak focusing:**

$$KN \xrightarrow{} KNwF + cut \xrightarrow{} KNwF$$
$$\text{cut-elimination}$$

$$\mathrm{cut} \frac{\Gamma\{P\} \quad \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}}$$

Completeness of focusing

Let $X \subseteq \{d, t, b, 4, 5\}$.
If A is provable in $KN + X^\diamond$, then any $\mathrm{pol}(A)$ is provable in $KNF + X^\diamond$.
Via cut-elimination:

$$\mathrm{cut} \frac{\Gamma\{P\} \quad \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}}$$

Problem:

$\rightarrow$ **Weak focusing:**

$$KN \longrightarrow KNwF + cut \longrightarrow KNwF \longrightarrow KNF$$

rules permutation

Completeness of focusing

Let $X \subseteq \{d, t, b, 4, 5\}$.
If A is provable in $KN + X^\diamond$, then any $\mathrm{pol}(A)$ is provable in $\mathsf{KNF} + X^\diamond$.
Via cut-elimination:

$$\mathrm{cut}\ \frac{\Gamma\{P\} \quad \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}}$$

Problem:

$\rightarrow$ **Weak focusing:**

$$KN \longrightarrow KNwF + cut \longrightarrow KNwF \longrightarrow KNF$$

$$\text{rules permutation}$$

$\rightarrow$ **Synthetic connectives:**

$$KN \longrightarrow KNF + cut \longrightarrow KNF$$

Completeness of focusing

Let $X \subseteq \{d, t, b, 4, 5\}$.
If A is provable in $KN + X^\diamond$, then any $\mathrm{pol}(A)$ is provable in KNF $+ X^\diamond$.
Via cut-elimination:

$$\mathrm{cut} \frac{\Gamma\{P\} \quad \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}}$$

Problem:

$\rightarrow$ **Weak focusing:**

$$KN \longrightarrow KNwF + cut \longrightarrow KNwF \longrightarrow KNF$$

rules permutation

$\rightarrow$ **Synthetic connectives:**

$$KN \longrightarrow KNS + cut \longrightarrow KNS$$

$$\cfrac{\mathsf{k}^\diamond \cfrac{\mathsf{id}\ \cfrac{}{[a, \bar{a}, \bar{b}, c]}}{\Diamond a, [\bar{a}, \bar{b}, c]} \qquad \mathsf{k}^\diamond \cfrac{\mathsf{id}\ \cfrac{}{[a, b, \bar{b}, c]}}{\Diamond a, [b, \bar{b}, c]}}{\cfrac{\mathsf{k}^\diamond \cfrac{\wedge \quad \Diamond a, [\bar{a} \wedge b, \bar{b}, c]}{\Diamond a, \Diamond(\bar{a} \wedge b), [\bar{b}, c]}}{\vee \quad \Diamond a, \Diamond(\bar{a} \wedge b), [\bar{b} \vee c]}}$$

$$\cfrac{\mathsf{pos} \cfrac{\mathsf{id}\ \cfrac{}{[a, \bar{a}, \bar{b}, c]}}{\Diamond a, [\bar{a}, \bar{b}, c]} \qquad \mathsf{pos} \cfrac{\mathsf{id}\ \cfrac{}{[a, b, \bar{b}, c]}}{\Diamond a, [b, \bar{b}, c]}}{\cfrac{\mathsf{pos} \quad \Diamond a, \Diamond(\bar{a} \wedge b), [\bar{b}, c]}{\mathsf{neg} \quad \Diamond a, \Diamond(\bar{a} \wedge b), [\bar{b} \vee c]}}$$

The focused nested system

Focused system KNF**:**

$$\text{dec } \frac{\Gamma\{P, \langle P \rangle\}}{\Gamma\{P\}} \qquad \Box \, \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \bar{\vee} \, \frac{\Gamma\{A, B\}}{\Gamma\{A \,\bar{\vee}\, B\}} \qquad \bar{\wedge} \, \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \,\bar{\wedge}\, B\}} \qquad \text{sto } \frac{\Gamma\{P\}}{\Gamma\{\uparrow P\}}$$

$$\text{id } \frac{}{\Gamma\{\bar{a}, \langle a \rangle\}} \qquad \mathsf{k}^{\diamond} \, \frac{\Gamma\{[\langle A \rangle, \Delta]\}}{\Gamma\{\langle \diamond A \rangle, [\Delta]\}} \qquad \overset{+}{\wedge} \, \frac{\Gamma\{\langle A \rangle\} \quad \Gamma\{\langle B \rangle\}}{\Gamma\{\langle A \,\overset{+}{\wedge}\, B \rangle\}} \qquad \overset{+}{\vee}_i \, \frac{\Gamma\{\langle A_i \rangle\}}{\Gamma\{\langle A_1 \,\overset{+}{\vee}\, A_2 \rangle\}} \quad \text{rel } \frac{\Gamma\{N\}}{\Gamma\{\langle \downarrow N \rangle\}}$$

Focused modal rules:

$$\mathsf{d}^{\diamond} \, \frac{\Gamma\{[\langle A \rangle]\}}{\Gamma\{\langle \diamond A \rangle\}} \qquad \mathsf{t}^{\diamond} \, \frac{\Gamma\{\langle A \rangle\}}{\Gamma\{\langle \diamond A \rangle\}} \qquad \mathsf{b}^{\diamond} \, \frac{\Gamma\{[\Delta], \langle A \rangle\}}{\Gamma\{[\Delta, \langle \diamond A \rangle]\}} \qquad 4^{\diamond} \, \frac{\Gamma\{[\langle \diamond A \rangle, \Delta]\}}{\Gamma\{\langle \diamond A \rangle, [\Delta]\}} \qquad 5^{\diamond} \, \frac{\Gamma\{\emptyset\}\{\langle \diamond A \rangle\}}{\Gamma\{\langle \diamond A \rangle\}\{\emptyset\}}$$

The synthetic nested system

Focused system KNF:

$$\text{dec } \frac{\Gamma\{P, \langle P\rangle\}}{\Gamma\{P\}} \qquad \Box \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \bar{\vee} \frac{\Gamma\{A, B\}}{\Gamma\{A \bar{\vee} B\}} \qquad \bar{\wedge} \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \bar{\wedge} B\}} \qquad \text{sto } \frac{\Gamma\{P\}}{\Gamma\{\uparrow P\}}$$

$$\text{id } \frac{}{\Gamma\{\bar{a}, \langle a\rangle\}} \qquad \text{k}^\diamond \frac{\Gamma\{[\langle A\rangle, \Delta]\}}{\Gamma\{\langle \diamond A\rangle, [\Delta]\}} \qquad \overset{+}{\wedge} \frac{\Gamma\{\langle A\rangle\} \quad \Gamma\{\langle B\rangle\}}{\Gamma\{\langle A \overset{+}{\wedge} B\rangle\}} \qquad \overset{+}{\vee}_i \frac{\Gamma\{\langle A_i\rangle\}}{\Gamma\{\langle A_1 \overset{+}{\vee} A_2\rangle\}} \qquad \text{rel } \frac{\Gamma\{N\}}{\Gamma\{\downarrow N\}}$$

Focused modal rules:

$$\text{d}^\diamond \frac{\Gamma\{[\langle A\rangle]\}}{\Gamma\{\langle \diamond A\rangle\}} \qquad \text{t}^\diamond \frac{\Gamma\{\langle A\rangle\}}{\Gamma\{\langle \diamond A\rangle\}} \qquad \text{b}^\diamond \frac{\Gamma\{[\Delta], \langle A\rangle\}}{\Gamma\{[\Delta, \langle \diamond A\rangle]\}} \qquad 4^\diamond \frac{\Gamma\{[\langle \diamond A\rangle, \Delta]\}}{\Gamma\{\langle \diamond A\rangle, [\Delta]\}} \qquad 5^\diamond \frac{\Gamma\{\emptyset\}\{\langle \diamond A\rangle\}}{\Gamma\{\langle \diamond A\rangle\}\{\emptyset\}}$$

The synthetic nested system

Synthetic system KNS:

$$\text{dec } \frac{\Gamma\{P, \langle P \rangle\}}{\Gamma\{P\}} \qquad\qquad \text{neg } \frac{\{\Gamma\{\Delta\}\}_{\Delta \preccurlyeq N}}{\Gamma\{N\}}$$

$$\text{id } \frac{}{\Gamma\{\bar{a}, \langle a \rangle\}} \quad \mathsf{k}^{\diamond} \frac{\Gamma\{[\langle A \rangle, \Delta]\}}{\Gamma\{\langle \diamond A \rangle, [\Delta]\}} \quad \overset{+}{\wedge} \frac{\Gamma\{\langle A \rangle\} \quad \Gamma\{\langle B \rangle\}}{\Gamma\{\langle A \overset{+}{\wedge} B \rangle\}} \quad \overset{+}{\vee}_i \frac{\Gamma\{\langle A_i \rangle\}}{\Gamma\{\langle A_1 \overset{+}{\vee} A_2 \rangle\}} \quad \text{rel } \frac{\Gamma\{N\}}{\Gamma\{\langle \downarrow N \rangle\}}$$

Focused modal rules:

$$\mathsf{d}^{\diamond} \frac{\Gamma\{[\langle A \rangle]\}}{\Gamma\{\langle \diamond A \rangle\}} \qquad \mathsf{t}^{\diamond} \frac{\Gamma\{\langle A \rangle\}}{\Gamma\{\langle \diamond A \rangle\}} \qquad \mathsf{b}^{\diamond} \frac{\Gamma\{[\Delta], \langle A \rangle\}}{\Gamma\{[\Delta, \langle \diamond A \rangle]\}} \qquad 4^{\diamond} \frac{\Gamma\{[\langle \diamond A \rangle, \Delta]\}}{\Gamma\{\langle \diamond A \rangle, [\Delta]\}} \qquad 5^{\diamond} \frac{\Gamma\{\emptyset\}\{\langle \diamond A \rangle\}}{\Gamma\{\langle \diamond A \rangle\}\{\emptyset\}}$$

Synthetic system KNS:

$$\text{pos } \frac{\Delta \preccurlyeq \bar{P} \quad \Gamma\{P, \langle\Delta\rangle\}}{\Gamma\{P\}} \qquad\qquad \text{neg } \frac{\{\Gamma\{\Delta\}\}_{\Delta \preccurlyeq N}}{\Gamma\{N\}}$$

$$\text{id } \frac{}{\Gamma\{\bar{a}, \langle a\rangle\}} \qquad \text{k}^\diamond \frac{\Gamma\{[\langle A\rangle, \Delta]\}}{\Gamma\{\langle\Diamond A\rangle, [\Delta]\}} \qquad \text{split } \frac{\Gamma\{\langle\Delta_1\rangle\} \quad \Gamma\{\langle\Delta_2\rangle\}}{\Gamma\{\langle\Delta_1, \Delta_2\rangle\}} \qquad \text{rel } \frac{\Gamma\{N\}}{\Gamma\{\langle\downarrow N\rangle\}}$$

Focused modal rules:

$$\text{d}^\diamond \frac{\Gamma\{[\langle A\rangle]\}}{\Gamma\{\langle\Diamond A\rangle\}} \qquad \text{t}^\diamond \frac{\Gamma\{\langle A\rangle\}}{\Gamma\{\langle\Diamond A\rangle\}} \qquad \text{b}^\diamond \frac{\Gamma\{[\Delta], \langle A\rangle\}}{\Gamma\{[\Delta, \langle\Diamond A\rangle]\}} \qquad 4^\diamond \frac{\Gamma\{[\langle\Diamond A\rangle, \Delta]\}}{\Gamma\{\langle\Diamond A\rangle, [\Delta]\}} \qquad 5^\diamond \frac{\Gamma\{\emptyset\}\{\langle\Diamond A\rangle\}}{\Gamma\{\langle\Diamond A\rangle\}\{\emptyset\}}$$

The synthetic nested system

Synthetic substructure matching:

$$\preccurlyeq\bar{\vee} \frac{\Gamma \preccurlyeq M \quad \Delta \preccurlyeq N}{\Gamma, \Delta \preccurlyeq M \bar{\vee} N} \qquad \preccurlyeq\bar{\wedge}_i \frac{\Gamma \preccurlyeq N_i}{\Gamma \preccurlyeq N_1 \bar{\wedge} N_2} \qquad \preccurlyeq\Box \frac{\Gamma \preccurlyeq N}{[\Gamma] \preccurlyeq \Box N} \qquad \preccurlyeq\uparrow \frac{}{P \preccurlyeq \uparrow P} \qquad \preccurlyeq\mathsf{id} \frac{}{\bar{a} \preccurlyeq \bar{a}}$$

Synthetic system KNS**:**

$$\mathsf{pos} \frac{\Delta \preccurlyeq \bar{P} \quad \Gamma\{P, \langle\Delta\rangle\}}{\Gamma\{P\}} \qquad\qquad \mathsf{neg} \frac{\{\Gamma\{\Delta\}\}_{\Delta \preccurlyeq N}}{\Gamma\{N\}}$$

$$\mathsf{id} \frac{}{\Gamma\{\bar{a}, \langle a\rangle\}} \qquad \mathsf{k}^\diamond \frac{\Gamma\{[\langle A\rangle, \Delta]\}}{\Gamma\{\langle\diamond A\rangle, [\Delta]\}} \qquad \mathsf{split} \frac{\Gamma\{\langle\Delta_1\rangle\} \quad \Gamma\{\langle\Delta_2\rangle\}}{\Gamma\{\langle\Delta_1, \Delta_2\rangle\}} \qquad \mathsf{rel} \frac{\Gamma\{N\}}{\Gamma\{\langle\downarrow N\rangle\}}$$

Focused modal rules:

$$\mathsf{d}^\diamond \frac{\Gamma\{[\langle A\rangle]\}}{\Gamma\{\langle\diamond A\rangle\}} \qquad \mathsf{t}^\diamond \frac{\Gamma\{\langle A\rangle\}}{\Gamma\{\langle\diamond A\rangle\}} \qquad \mathsf{b}^\diamond \frac{\Gamma\{[\Delta], \langle A\rangle\}}{\Gamma\{[\Delta, \langle\diamond A\rangle]\}} \qquad 4^\diamond \frac{\Gamma\{[\langle\diamond A\rangle, \Delta]\}}{\Gamma\{\langle\diamond A\rangle, [\Delta]\}} \qquad 5^\diamond \frac{\Gamma\{\emptyset\}\{\langle\diamond A\rangle\}}{\Gamma\{\langle\diamond A\rangle\}\{\emptyset\}}$$

The synthetic nested system

Synthetic substructure matching:

$$\preccurlyeq\bar{\vee}\ \frac{\Gamma\preccurlyeq M \quad \Delta\preccurlyeq N}{\Gamma,\Delta\preccurlyeq M\ \bar{\vee}\ N} \quad \preccurlyeq\bar{\wedge}_i\ \frac{\Gamma\preccurlyeq N_i}{\Gamma\preccurlyeq N_1\ \bar{\wedge}\ N_2} \quad \preccurlyeq\Box\ \frac{\Gamma\preccurlyeq N}{[\Gamma]\preccurlyeq\Box N} \quad \preccurlyeq\uparrow\ \frac{}{P\preccurlyeq\uparrow P} \quad \preccurlyeq\mathsf{id}\ \frac{}{\bar{a}\preccurlyeq\bar{a}}$$

Synthetic system KNS**:**

$$\mathsf{pos}\ \frac{\Delta\preccurlyeq\bar{P}\quad\Gamma\{P,\langle\Delta\rangle\}}{\Gamma\{P\}} \qquad\qquad \mathsf{neg}\ \frac{\{\Gamma\{\Delta\}\}_{\Delta\preccurlyeq N}}{\Gamma\{N\}}$$

$$\mathsf{id}\ \frac{}{\Gamma\{\bar{a},\langle\bar{a}\rangle\}} \qquad \mathsf{k}^\diamond\ \frac{\Gamma\{[\langle A\rangle,\Delta]\}}{\Gamma\{\langle\diamond A\rangle,[\Delta]\}} \qquad \mathsf{split}\ \frac{\Gamma\{\langle\Delta_1\rangle\}\quad\Gamma\{\langle\Delta_2\rangle\}}{\Gamma\{\langle\Delta_1,\Delta_2\rangle\}} \qquad \mathsf{rel}\ \frac{\Gamma\{\bar{P}\}}{\Gamma\{\langle P\rangle\}}$$

Focused modal rules:

$$\mathsf{d}^\diamond\ \frac{\Gamma\{[\langle A\rangle]\}}{\Gamma\{\langle\diamond A\rangle\}} \qquad \mathsf{t}^\diamond\ \frac{\Gamma\{\langle A\rangle\}}{\Gamma\{\langle\diamond A\rangle\}} \qquad \mathsf{b}^\diamond\ \frac{\Gamma\{[\Delta],\langle A\rangle\}}{\Gamma\{[\Delta,\langle\diamond A\rangle]\}} \qquad 4^\diamond\ \frac{\Gamma\{[\langle\diamond A\rangle,\Delta]\}}{\Gamma\{\langle\diamond A\rangle,[\Delta]\}} \qquad 5^\diamond\ \frac{\Gamma\{\emptyset\}\{\langle\diamond A\rangle\}}{\Gamma\{\langle\diamond A\rangle\}\{\emptyset\}}$$

The synthetic nested system

Synthetic substructure matching:

$$\preccurlyeq\bar{\vee} \; \frac{\Gamma \preccurlyeq M \quad \Delta \preccurlyeq N}{\Gamma, \Delta \preccurlyeq M \,\bar{\vee}\, N} \qquad \preccurlyeq\bar{\wedge}_i \; \frac{\Gamma \preccurlyeq N_i}{\Gamma \preccurlyeq N_1 \,\bar{\wedge}\, N_2} \qquad \preccurlyeq\Box \; \frac{\Gamma \preccurlyeq N}{[\Gamma] \preccurlyeq \Box N} \qquad \preccurlyeq\uparrow \; \frac{}{P \preccurlyeq \uparrow P} \qquad \preccurlyeq\mathsf{id} \; \frac{}{\bar{a} \preccurlyeq \bar{a}}$$

Synthetic system KNS**:**

$$\mathsf{pos} \; \frac{\Delta \preccurlyeq \bar{P} \quad \Gamma\{P, \langle\Delta\rangle\}}{\Gamma\{P\}} \qquad\qquad \mathsf{neg} \; \frac{\{\Gamma\{\Delta\}\}_{\Delta\preccurlyeq N}}{\Gamma\{N\}}$$

$$\mathsf{id} \; \frac{}{\Gamma\{\bar{a}, \langle\bar{a}\rangle\}} \qquad \mathsf{k}^{\langle\rangle} \; \frac{\Gamma\{[\langle\Delta\rangle, \Omega]\}}{\Gamma\{\langle[\Delta]\rangle, [\Omega]\}} \qquad \mathsf{split} \; \frac{\Gamma\{\langle\Delta_1\rangle\} \quad \Gamma\{\langle\Delta_2\rangle\}}{\Gamma\{\langle\Delta_1, \Delta_2\rangle\}} \qquad \mathsf{rel} \; \frac{\Gamma\{\bar{P}\}}{\Gamma\{\langle P\rangle\}}$$

Focused modal rules:

$$\mathsf{d}^{\diamond} \; \frac{\Gamma\{[\langle A\rangle]\}}{\Gamma\{\langle\diamond A\rangle\}} \qquad \mathsf{t}^{\diamond} \; \frac{\Gamma\{\langle A\rangle\}}{\Gamma\{\langle\diamond A\rangle\}} \qquad \mathsf{b}^{\diamond} \; \frac{\Gamma\{[\Delta], \langle A\rangle\}}{\Gamma\{[\Delta, \langle\diamond A\rangle]\}} \qquad \mathsf{4}^{\diamond} \; \frac{\Gamma\{[\langle\diamond A\rangle, \Delta]\}}{\Gamma\{\langle\diamond A\rangle, [\Delta]\}} \qquad \mathsf{5}^{\diamond} \; \frac{\Gamma\{\emptyset\}\{\langle\diamond A\rangle\}}{\Gamma\{\langle\diamond A\rangle\}\{\emptyset\}}$$

The synthetic nested system

Synthetic substructure matching:

$$\preccurlyeq\bar{\vee}\ \frac{\Gamma \preccurlyeq M \quad \Delta \preccurlyeq N}{\Gamma, \Delta \preccurlyeq M \bar{\vee} N} \quad \preccurlyeq\bar{\wedge}_i\ \frac{\Gamma \preccurlyeq N_i}{\Gamma \preccurlyeq N_1 \bar{\wedge} N_2} \quad \preccurlyeq\Box\ \frac{\Gamma \preccurlyeq N}{[\Gamma] \preccurlyeq \Box N} \quad \preccurlyeq\uparrow\ \frac{}{P \preccurlyeq \uparrow P} \quad \preccurlyeq\text{id}\ \frac{}{\bar{a} \preccurlyeq \bar{a}}$$

Synthetic system KNS:

$$\text{pos}\ \frac{\Delta \preccurlyeq \bar{P} \quad \Gamma\{P, \langle\Delta\rangle\}}{\Gamma\{P\}} \qquad\qquad \text{neg}\ \frac{\{\Gamma\{\Delta\}\}_{\Delta \preccurlyeq N}}{\Gamma\{N\}}$$

$$\text{id}\ \frac{}{\Gamma\{\bar{a}, \langle\bar{a}\rangle\}} \qquad \text{k}^{\Diamond}\ \frac{\Gamma\{[\langle\Delta\rangle, \Omega]\}}{\Gamma\{\langle[\Delta]\rangle, [\Omega]\}} \qquad \text{split}\ \frac{\Gamma\{\langle\Delta_1\rangle\} \quad \Gamma\{\langle\Delta_2\rangle\}}{\Gamma\{\langle\Delta_1, \Delta_2\rangle\}} \qquad \text{rel}\ \frac{\Gamma\{\bar{P}\}}{\Gamma\{\langle P\rangle\}}$$

Synthetic modal rules:

$$\text{d}^{\Diamond}\ \frac{\Gamma\{[\langle\Delta\rangle]\}}{\Gamma\{\langle[\Delta]\rangle\}} \qquad \text{t}^{\Diamond}\ \frac{\Gamma\{\langle\Delta\rangle\}}{\Gamma\{\langle[\Delta]\rangle\}} \qquad \text{b}^{\Diamond}\ \frac{\Gamma\{[\Omega, \langle\Delta\rangle]\}}{\Gamma\{[\Omega, \langle[\Delta]\rangle]\}} \qquad 4^{\Diamond}\ \frac{\Gamma\{[\Omega, \langle[\Delta]\rangle]\}}{\Gamma\{[\Omega], \langle[\Delta]\rangle\}} \qquad 5^{\Diamond}\ \frac{\Gamma\{\langle[\Delta]\rangle\}\{\emptyset\}}{\Gamma\{\emptyset\}\{\langle[\Delta]\rangle\}}$$

In action...

Synthetic connectives: $\text{neg}\ \dfrac{\{\Gamma\{\Delta\}\}_{\Delta\preccurlyeq N}}{\Gamma\{N\}}$ and $\text{pos}\ \dfrac{\Delta\preccurlyeq\bar{P}\quad\Gamma\{P,\langle\Delta\rangle\}}{\Gamma\{P\}}$

Structural modal rules : distinct modal phase and action on substructures

$$
\text{neg}\ \cfrac{\text{pos}\ \cfrac{\text{k}^{\Diamond}\ \cfrac{\text{id}\ \overline{\Diamond(a\overset{\cdot}{\vee}b),[\langle\bar{a}\rangle,\bar{a}]}}{\Diamond(a\overset{\cdot}{\vee}b),\langle\bar{a}\rangle,[\bar{a}]}}{\Diamond(a\overset{\cdot}{\vee}b),[\bar{a}]}\quad\text{pos}\ \cfrac{\text{k}^{\Diamond}\ \cfrac{\text{id}\ \overline{\Diamond(a\overset{\cdot}{\vee}b),[\langle\bar{b}\rangle,]}}{\Diamond(a\overset{\cdot}{\vee}b),\langle\bar{b}\rangle,[\bar{b}]}}{\Diamond(a\overset{\cdot}{\vee}b),[\bar{b}]}}{\Diamond(a\overset{\cdot}{\vee}b),\Box(\bar{a}\bar{\wedge}\bar{b})}
$$

$$
\preccurlyeq\Box\ \cfrac{\preccurlyeq\bar{\wedge}\ \cfrac{\preccurlyeq\text{id}\ \overline{\bar{a}\preccurlyeq\bar{a}}}{\bar{a}\preccurlyeq\bar{a}\bar{\wedge}\bar{b}}}{[\bar{a}]\preccurlyeq\Box(\bar{a}\bar{\wedge}\bar{b})}\qquad\qquad\preccurlyeq\Box\ \cfrac{\preccurlyeq\bar{\wedge}\ \cfrac{\preccurlyeq\text{id}\ \overline{\bar{b}\preccurlyeq\bar{a}}}{\bar{b}\preccurlyeq\bar{a}\bar{\wedge}\bar{b}}}{[\bar{b}]\preccurlyeq\Box(\bar{a}\bar{\wedge}\bar{b})}
$$

Synthetic permutation

Synthetic permutation

Synthetic permutation

$$KN \longrightarrow KNS + cut \longrightarrow KNS$$
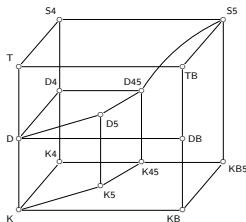
Cut-elimination

Theorem Let $X \subseteq \{d, t, b, 4, 5\}$ be 45-closed.
If a sequent Γ is provable in $\text{KNF} + X^\diamond + \text{Cut}$, then it is also provable in $\text{KNF} + X^\diamond$.

$$\text{Cut} = \left\{ \text{cut}_1 \frac{\Gamma\{P\} \quad \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}}; \text{cut}_2 \frac{\Gamma\{\langle P \rangle\} \quad \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}}; \text{cut}_3 \frac{\Gamma\{\langle Q \rangle\}\{P\} \quad \Gamma\{\emptyset\}\{\bar{P}\}}{\Gamma\{\langle Q \rangle\}\{\emptyset\}} \right\}$$

$$clo(X) = \begin{cases} X \cup \{4\} & \text{if } \{b, 5\} \subseteq X \text{ or if } \{t, 5\} \subseteq X \\ X \cup \{5\} & \text{if } \{b, 4\} \subseteq X \\ X & \text{otherwise} \end{cases}$$

Cut-elimination proof

$$
\mathrm{cut}\;\dfrac{
\dfrac{
\mathrm{b}^{\diamond}\;\dfrac{
\Box\;\dfrac{
\diamond\;\dfrac{
\mathrm{id}\;\dfrac{}{[\bar{a}, a], [\diamond a]}
}{[\bar{a}], \diamond a, [\diamond a]}
}{\Box\bar{a}, \diamond a, [\diamond a]}
}{\Box\bar{a}, [\diamond\diamond a, \diamond a]}
\qquad
\Box\;\dfrac{
4^{\diamond}\;\dfrac{
\Box\;\dfrac{
\diamond\;\dfrac{
\mathrm{id}\;\dfrac{}{\Box\bar{a}, [[[\bar{a}, a]]]}
}{\Box\bar{a}, [[[\bar{a}], \diamond a]]}
}{\Box\bar{a}, [[\Box\bar{a}, \diamond a]]}
}{\Box\bar{a}, [[\Box\bar{a}], \diamond a]}
}{\Box\bar{a}, [\Box\Box\bar{a}, \diamond a]}
}{\Box\bar{a}, [\diamond a]}
\qquad\rightsquigarrow\qquad
5^{\diamond}\;\dfrac{
\Box\;\dfrac{
\diamond\;\dfrac{
\mathrm{id}\;\dfrac{}{[\bar{a}, a], [\ ]}
}{[\bar{a}], \diamond a, [\ ]}
}{\Box\bar{a}, \diamond a, [\ ]}
}{\Box\bar{a}, [\diamond a]}
$$

Completeness proof

KN

$$\begin{array}{l}
\text{id}\ \dfrac{\phantom{[\bar{a}, a], [\ ]}}{[\bar{a}, a], [\ ]} \\[4pt]
\diamond\ \dfrac{}{[\bar{a}], \diamond a, [\ ]} \\[4pt]
\Box\ \dfrac{}{\Box\bar{a}, \diamond a, [\ ]} \\[4pt]
5^{\circ}\ \dfrac{}{\Box\bar{a}, [\diamond a]} \\[4pt]
\Box\ \dfrac{}{\Box\bar{a}, \Box\diamond a}
\end{array}$$

Completeness proof

simulation

$$\text{KN} \xrightarrow{\hspace{3cm}} \text{KNwF} + \text{cut}$$

$$
\text{id} \; \dfrac{\overline{\phantom{[\bar{a}, a], [\;]}}}{
\diamond \; \dfrac{[\bar{a}, a], [\;]}{
\square \; \dfrac{[\bar{a}], \diamond a, [\;]}{
5^{\diamond} \; \dfrac{\square \bar{a}, \diamond a, [\;]}{
\square \; \dfrac{\square \bar{a}, [\diamond a]}{
\square \bar{a}, \square \diamond a}}}}}
\tag{1}
$$

Completeness proof

$$\text{KN} \longrightarrow \text{KNwF} + \text{cut} \xrightarrow{\text{cut-elimination}} \text{KNwF}$$

Completeness proof

rules permutation

$$\text{KN} \longrightarrow \text{KNwF} + \text{cut} \longrightarrow \text{KNwF} \longrightarrow \text{KNF}$$

$$\text{id} \frac{}{[\bar{a}, \langle a \rangle], [\Diamond a]}$$
$$\Diamond \frac{}{[\bar{a}], \langle \Diamond a \rangle, [\Diamond a]}$$
$$\Box \frac{}{\Box \bar{a}, \langle \Diamond a \rangle, [\Diamond a]}$$
$$5\Diamond \frac{}{\Box \bar{a}, [\Diamond a, \langle \Diamond a \rangle]}$$
$$\text{dec} \frac{}{\Box \bar{a}, [\Diamond a]}$$
$$\text{sto} \frac{}{\Box \bar{a}, [\uparrow \Diamond a]}$$
$$\Box \frac{}{\Box \bar{a}, \Box \uparrow \Diamond a}$$

$$\text{id} \frac{}{[\bar{a}, \langle a \rangle], [\Diamond a]}$$
$$\Diamond \frac{}{[\bar{a}], \langle \Diamond a \rangle, [\Diamond a]}$$
$$5\Diamond \frac{}{[\bar{a}], [\Diamond a, \langle \Diamond a \rangle]}$$
$$\text{dec} \frac{}{[\bar{a}], [\Diamond a]}$$
$$\text{sto} \frac{}{[\bar{a}], [\uparrow \Diamond a]}$$
$$\Box \frac{}{\Box \bar{a}, [\uparrow \Diamond a]}$$
$$\Box \frac{}{\Box \bar{a}, \Box \uparrow \Diamond a}$$

In action…

- neg and pos : synthetic connectives
- structural modal rules : modal phase / action on substructures