

# Focused Nested Sequents

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→ **Focusing** :

distillation process for proofs in linear, intuitionistic and classical logic

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→ **Nested sequents** : modular proof systems for modal logics

# Classical modal logic

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$$k: \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$$

**Rules:** modus ponens:  $\frac{A \quad A \rightarrow B}{B}$       necessitation:  $\frac{A}{\Box A}$

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**The S5-cube:**

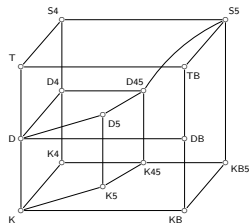
d:  $\Box A \rightarrow \Diamond A$

t:  $A \rightarrow \Diamond A$

b:  $A \rightarrow \Box \Diamond A$

4:  $\Diamond \Diamond A \rightarrow \Diamond A$

5:  $\Diamond A \rightarrow \Box \Diamond A$



# Nested sequents

**Sequent:**

$$\Gamma ::= A_1, \dots, A_m$$

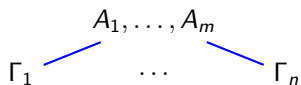
$$fm(\Gamma) = A_1 \vee \dots \vee A_m$$



# Nested sequents

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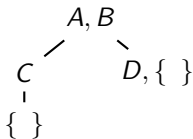
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**Sequent context:**  $\Gamma \{ \} \{ \} = A, B, [C, [\{ \}]], [D, \{ \}]$



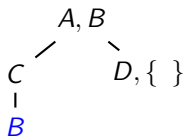
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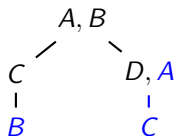
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[Kashima, 1994], [Brünnler, 2009], [Poggiolesi, 2009]

# The standard nested system

**Formulas:**  $A ::= a \mid \bar{a} \mid A \wedge A \mid A \vee A \mid \Box A \mid \Diamond A$

**System KN:**

$$\begin{array}{l} \text{cont} \frac{\Gamma\{A, A\}}{\Gamma\{A\}} \quad \Box \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \quad \vee \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}} \\ \text{id} \frac{}{\Gamma\{a, \bar{a}\}} \quad k^\diamond \frac{\Gamma\{[A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} \quad \wedge \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \\ k: \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B) \end{array}$$

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**Modal rules:**

$$d^\diamond \frac{\Gamma\{[A]\}}{\Gamma\{\Diamond A\}}$$

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d:  $\Box A \rightarrow \Diamond A$

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[Brünnler, 2009]

# Focusing and polarities

$$\begin{array}{c} \text{id} \frac{}{[a, \bar{a}, \bar{b}, c]} \quad \text{id} \frac{}{[b, \bar{a}, \bar{b}, c]} \\ \wedge \frac{}{[a, \bar{a} \wedge b, \bar{b}, c]} \\ \vee \frac{}{[a, \bar{a} \wedge b, \bar{b} \vee c]} \\ \text{k}^\diamond \frac{}{\diamond a, [\bar{a} \wedge b, \bar{b} \vee c]} \\ \text{k}^\diamond \frac{}{\diamond a, \diamond(\bar{a} \wedge b), [\bar{b} \vee c]} \end{array}$$

# Focusing and polarities

**Polarities:**      invertible rules      :      negative connectives  
                  non-invertible rules      :      Positive connectives

[Andreoli, 1992], [Laurent, 2004]



# Focusing and polarities

**Weak focusing:** For any subproof  $\frac{\pi'}{\Gamma\{P\}}$  the **only positive** rules between two rules decomposing  $P$  are rules decomposing  $P$ .

$$\frac{\frac{\text{id} \frac{\overline{[a, \bar{a}, \bar{b}, c]}}{\diamond a, [\bar{a}, \bar{b}, c]}}{\wedge} \quad \frac{\text{id} \frac{\overline{[a, b, \bar{b}, c]}}{\diamond a, [b, \bar{b}, c]}}{\wedge}}{\vee} \frac{\frac{\diamond a, [\bar{a} \wedge b, \bar{b}, c]}{\diamond a, [\bar{a} \wedge b, \bar{b} \vee c]}}{\text{k}^\diamond} \frac{\diamond a, \diamond(\bar{a} \wedge b), [\bar{b} \vee c]}{\text{k}^\diamond}}$$

[Andreoli, 1992], [Laurent, 2004]

# Focusing and polarities

**Strong focusing:** For any subproof  $\frac{\pi'}{\Gamma\{P\}}$  the **only** rules between two rules decomposing  $P$  are rules decomposing  $P$ .

$$\begin{array}{c}
 \text{id} \frac{\overline{[a, \bar{a}, \bar{b}, c]}}{\overline{[a, \bar{a}, \bar{b}, c]}} \quad \text{id} \frac{\overline{[a, b, \bar{b}, c]}}{\overline{[a, b, \bar{b}, c]}} \\
 \text{k}^\diamond \frac{\overline{\diamond a, [\bar{a}, \bar{b}, c]}}{\overline{\diamond a, [\bar{a}, \bar{b}, c]}} \quad \text{k}^\diamond \frac{\overline{\diamond a, [b, \bar{b}, c]}}{\overline{\diamond a, [b, \bar{b}, c]}} \\
 \vee \frac{\overline{\diamond a, [\bar{a}, \bar{b} \vee c]}}{\overline{\diamond a, [\bar{a}, \bar{b} \vee c]}} \quad \vee \frac{\overline{\diamond a, [b, \bar{b} \vee c]}}{\overline{\diamond a, [b, \bar{b} \vee c]}} \\
 \wedge \frac{\overline{\diamond a, [\bar{a} \wedge b, \bar{b} \vee c]}}{\overline{\diamond a, [\bar{a} \wedge b, \bar{b} \vee c]}} \\
 \text{k}^\diamond \frac{\overline{\diamond a, \diamond(\bar{a} \wedge b), [\bar{b} \vee c]}}{\overline{\diamond a, \diamond(\bar{a} \wedge b), [\bar{b} \vee c]}}
 \end{array}$$

[Andreoli, 1992], [Laurent, 2004]

# Focusing and polarities

**Inversion:** For any subproof  $\frac{\pi'}{\Gamma\{N\}}$  the last rule is negative.

$$\frac{\frac{\text{id} \frac{\overline{[a, \bar{a}, \bar{b}, c]}}{[a, \bar{a}, \bar{b}, c]}}{\text{k}^\diamond \frac{\overline{\diamond a, [\bar{a}, \bar{b}, c]}}{\diamond a, [\bar{a}, \bar{b}, c]}} \quad \frac{\text{id} \frac{\overline{[a, b, \bar{b}, c]}}{[a, b, \bar{b}, c]}}{\text{k}^\diamond \frac{\overline{\diamond a, [b, \bar{b}, c]}}{\diamond a, [b, \bar{b}, c]}}}{\wedge \frac{\overline{\diamond a, [\bar{a} \wedge b, \bar{b}, c]}}{\text{k}^\diamond \frac{\overline{\diamond a, \diamond(\bar{a} \wedge b), [\bar{b}, c]}}{\diamond a, \diamond(\bar{a} \wedge b), [\bar{b}, c]}}} \quad \vee \frac{\overline{\diamond a, \diamond(\bar{a} \wedge b), [\bar{b} \vee c]}}{\diamond a, \diamond(\bar{a} \wedge b), [\bar{b} \vee c]}}$$

[Andreoli, 1992], [Laurent, 2004]

# The standard nested system

**Formulas:**  $A ::= a \mid \bar{a} \mid A \wedge A \mid A \vee A \mid \Box A \mid \Diamond A$

**System KN:**

$$\begin{array}{l} \text{cont} \frac{\Gamma\{A, A\}}{\Gamma\{A\}} \quad \Box \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \quad \vee \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}} \\ \text{id} \frac{}{\Gamma\{a, \bar{a}\}} \quad \text{k}^\diamond \frac{\Gamma\{[A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} \quad \wedge \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \end{array}$$

**Modal rules:**

$$\text{d}^\diamond \frac{\Gamma\{[A]\}}{\Gamma\{\Diamond A\}} \quad \text{t}^\diamond \frac{\Gamma\{A\}}{\Gamma\{\Diamond A\}} \quad \text{b}^\diamond \frac{\Gamma\{[\Delta], A\}}{\Gamma\{[\Delta, \Diamond A]\}} \quad \text{4}^\diamond \frac{\Gamma\{[\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} \quad \text{5}^\diamond \frac{\Gamma\{\emptyset\}\{\Diamond A\}}{\Gamma\{\Diamond A\}\{\emptyset\}}$$

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# The **focused** nested system

**Polarized formulas:**

$$\begin{array}{l} P ::= a \mid \downarrow N \mid \diamond P \mid P \wedge P \\ N ::= \bar{a} \mid \uparrow P \mid \square N \mid N \vee N \end{array}$$

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$$\begin{array}{l} P ::= a \mid \downarrow N \mid \diamond P \mid P \wedge P \\ N ::= \bar{a} \mid \uparrow P \mid \square N \mid N \vee N \end{array}$$

**Focused system KNF:**

$$\begin{array}{l} \text{dec} \frac{\Gamma\{P, \langle P \rangle\}}{\Gamma\{P\}} \quad \square \frac{\Gamma\{[A]\}}{\Gamma\{\square A\}} \quad \vee \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}} \\ \text{id} \frac{}{\Gamma\{\bar{a}, \langle a \rangle\}} \quad k^\diamond \frac{\Gamma\{\langle A \rangle, \Delta\}}{\Gamma\{\langle \diamond A \rangle, [\Delta]\}} \quad \wedge \frac{\Gamma\{\langle A \rangle\} \quad \Gamma\{\langle B \rangle\}}{\Gamma\{\langle A \wedge B \rangle\}} \end{array}$$

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$$\begin{array}{l} \text{d}^\diamond \frac{\Gamma\{[A]\}}{\Gamma\{\langle \diamond A \rangle\}} \quad \text{t}^\diamond \frac{\Gamma\{A\}}{\Gamma\{\langle \diamond A \rangle\}} \quad \text{b}^\diamond \frac{\Gamma\{[\Delta], A\}}{\Gamma\{[\Delta], \langle \diamond A \rangle\}} \quad 4^\diamond \frac{\Gamma\{[\langle \diamond A \rangle, \Delta]\}}{\Gamma\{\langle \diamond A \rangle, [\Delta]\}} \quad 5^\diamond \frac{\Gamma\{\emptyset\}\langle \diamond A \rangle}{\Gamma\{\langle \diamond A \rangle\}\emptyset} \end{array}$$



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# The **focused** nested system

**Polarized formulas:**

$$\begin{array}{l}
 P ::= a \mid \downarrow N \mid \diamond P \mid P \dot{\wedge} P \mid P \dot{\vee} P \\
 N ::= \bar{a} \mid \uparrow P \mid \square N \mid N \bar{\vee} N \mid N \bar{\wedge} N
 \end{array}$$

**Focused system KNF:**

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 \text{dec} \frac{\Gamma\{P, \langle P \rangle\}}{\Gamma\{P\}} \quad \square \frac{\Gamma\{[A]\}}{\Gamma\{\square A\}} \quad \bar{\vee} \frac{\Gamma\{A, B\}}{\Gamma\{A \bar{\vee} B\}} \quad \bar{\wedge} \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \bar{\wedge} B\}} \quad \text{sto} \frac{\Gamma\{P\}}{\Gamma\{\uparrow P\}} \\
 \text{id} \frac{}{\Gamma\{\bar{a}, \langle a \rangle\}} \quad k^\diamond \frac{\Gamma\{\langle A \rangle, [\Delta]\}}{\Gamma\{\langle \diamond A \rangle, [\Delta]\}} \quad \dot{\wedge} \frac{\Gamma\{\langle A \rangle\} \quad \Gamma\{\langle B \rangle\}}{\Gamma\{\langle A \dot{\wedge} B \rangle\}} \quad \dot{\vee}_i \frac{\Gamma\{\langle A_i \rangle\}}{\Gamma\{\langle A_1 \dot{\vee} A_2 \rangle\}} \quad \text{rel} \frac{\Gamma\{N\}}{\Gamma\{\langle \downarrow N \rangle\}}
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 \end{array}$$

**Focused modal rules:**

$$\begin{array}{c}
 d^\diamond \frac{\Gamma\{\langle A \rangle\}}{\Gamma\{\langle \diamond A \rangle\}} \quad t^\diamond \frac{\Gamma\{\langle A \rangle\}}{\Gamma\{\langle \diamond A \rangle\}} \quad b^\diamond \frac{\Gamma\{[\Delta], \langle A \rangle\}}{\Gamma\{[\Delta], \langle \diamond A \rangle\}} \quad 4^\diamond \frac{\Gamma\{\langle \diamond A \rangle, \Delta\}}{\Gamma\{\langle \diamond A \rangle, [\Delta]\}} \quad 5^\diamond \frac{\Gamma\{\emptyset\}\langle \diamond A \rangle\}}{\Gamma\{\langle \diamond A \rangle\}\emptyset\}}
 \end{array}$$

# Completeness of focusing

Let  $X \subseteq \{d, t, b, 4, 5\}$ .

If  $A$  is provable in  $\text{KN} + X^\diamond$ , then any  $\text{pol}(A)$  is provable in  $\text{KNF} + X^\diamond$ .

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**Via cut-elimination:**

$$\text{cut} \frac{\Gamma\{P\} \quad \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}}$$

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simulation  
 $\text{KN} \longrightarrow \text{KNF} + \text{cut}$

$$\text{cut} \frac{\Gamma\{P\} \quad \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}}$$

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**Via cut-elimination:**

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$$\text{KN} \longrightarrow \text{KNF} + \text{cut} \xrightarrow{\text{cut-elimination}} \text{KNF}$$

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**Problem:**

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**Problem:**

→ **Weak focusing:**

$\text{KN} \xrightarrow{\text{simulation}} \text{KNwF} + \text{cut}$

$$\text{cut} \frac{\Gamma\{P\} \quad \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}}$$

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If  $A$  is provable in  $\text{KN} + X^\diamond$ , then any  $\text{pol}(A)$  is provable in  $\text{KNF} + X^\diamond$ .

**Via cut-elimination:**

$$\text{cut} \frac{\Gamma\{P\} \quad \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}}$$

**Problem:**

→ **Weak focusing:**

$$\text{KN} \longrightarrow \text{KNwF} + \text{cut} \xrightarrow{\text{cut-elimination}} \text{KNwF}$$

$$\text{cut} \frac{\Gamma\{P\} \quad \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}}$$

# Completeness of focusing

Let  $X \subseteq \{d, t, b, 4, 5\}$ .

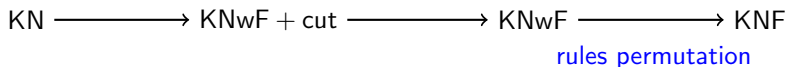
If  $A$  is provable in  $KN + X^\diamond$ , then any  $\text{pol}(A)$  is provable in  $KNF + X^\diamond$ .

**Via cut-elimination:**

$$\text{cut} \frac{\Gamma\{P\} \quad \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}}$$

**Problem:**

→ **Weak focusing:**



# Completeness of focusing

Let  $X \subseteq \{d, t, b, 4, 5\}$ .

If  $A$  is provable in  $KN + X^\diamond$ , then any  $\text{pol}(A)$  is provable in  $KNF + X^\diamond$ .

**Via cut-elimination:**

$$\text{cut} \frac{\Gamma\{P\} \quad \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}}$$

**Problem:**

→ **Weak focusing:**

$$KN \longrightarrow KNwF + \text{cut} \longrightarrow KNwF \xrightarrow{\text{rules permutation}} KNF$$

→ **Synthetic connectives:**

$$KN \longrightarrow KNF + \text{cut} \longrightarrow KNF$$

# Completeness of focusing

Let  $X \subseteq \{d, t, b, 4, 5\}$ .

If  $A$  is provable in  $KN + X^\diamond$ , then any  $\text{pol}(A)$  is provable in  $KNF + X^\diamond$ .

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$$\text{cut} \frac{\Gamma\{P\} \quad \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}}$$

**Problem:**

→ **Weak focusing:**

$$KN \longrightarrow KNwF + \text{cut} \longrightarrow KNwF \xrightarrow{\text{rules permutation}} KNF$$

→ **Synthetic connectives:**

$$KN \longrightarrow KNS + \text{cut} \longrightarrow KNS$$

$$\begin{array}{c}
\text{id} \frac{}{[a, \bar{a}, \bar{b}, c]} \quad \text{id} \frac{}{[a, b, \bar{b}, c]} \\
\text{k}^\diamond \frac{}{\diamond a, [\bar{a}, \bar{b}, c]} \quad \text{k}^\diamond \frac{}{\diamond a, [b, \bar{b}, c]} \\
\wedge \frac{}{\diamond a, [\bar{a} \wedge b, \bar{b}, c]} \\
\text{k}^\diamond \frac{}{\diamond a, \diamond(\bar{a} \wedge b), [\bar{b}, c]} \\
\vee \frac{}{\diamond a, \diamond(\bar{a} \wedge b), [\bar{b} \vee c]}
\end{array}$$

$$\begin{array}{c}
\text{id} \frac{}{[a, \bar{a}, \bar{b}, c]} \quad \text{id} \frac{}{[a, b, \bar{b}, c]} \\
\text{pos} \frac{}{\diamond a, [\bar{a}, \bar{b}, c]} \quad \text{pos} \frac{}{\diamond a, [b, \bar{b}, c]} \\
\text{pos} \frac{}{\diamond a, \diamond(\bar{a} \wedge b), [\bar{b}, c]} \\
\text{neg} \frac{}{\diamond a, \diamond(\bar{a} \wedge b), [\bar{b} \vee c]}
\end{array}$$

# The focused nested system

## Focused system KNF:

$$\begin{array}{cccccc} \text{dec} \frac{\Gamma\{P, \langle P \rangle\}}{\Gamma\{P\}} & \square \frac{\Gamma\{[A]\}}{\Gamma\{\square A\}} & \bar{\vee} \frac{\Gamma\{A, B\}}{\Gamma\{A \bar{\vee} B\}} & \bar{\wedge} \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \bar{\wedge} B\}} & \text{sto} \frac{\Gamma\{P\}}{\Gamma\{\uparrow P\}} \\ \text{id} \frac{}{\Gamma\{\bar{a}, \langle a \rangle\}} & k^\circ \frac{\Gamma\{[\langle A \rangle, \Delta]\}}{\Gamma\{\langle \diamond A \rangle, [\Delta]\}} & \hat{\wedge} \frac{\Gamma\{\langle A \rangle\} \quad \Gamma\{\langle B \rangle\}}{\Gamma\{\langle A \hat{\wedge} B \rangle\}} & \dot{\vee}_i \frac{\Gamma\{\langle A_i \rangle\}}{\Gamma\{\langle A_1 \dot{\vee} A_2 \rangle\}} & \text{rel} \frac{\Gamma\{N\}}{\Gamma\{\langle \downarrow N \rangle\}} \end{array}$$

## Focused modal rules:

$$\begin{array}{ccccc} d^\circ \frac{\Gamma\{[\langle A \rangle]\}}{\Gamma\{\langle \diamond A \rangle\}} & t^\circ \frac{\Gamma\{\langle A \rangle\}}{\Gamma\{\langle \diamond A \rangle\}} & b^\circ \frac{\Gamma\{[\Delta], \langle A \rangle\}}{\Gamma\{[\Delta], \langle \diamond A \rangle\}} & 4^\circ \frac{\Gamma\{[\langle \diamond A \rangle, \Delta]\}}{\Gamma\{\langle \diamond A \rangle, [\Delta]\}} & 5^\circ \frac{\Gamma\{\emptyset\}\{\langle \diamond A \rangle\}}{\Gamma\{\langle \diamond A \rangle\}\{\emptyset\}} \end{array}$$

# The synthetic nested system

## Focused system KNF:

$$\begin{array}{cccccc} \text{dec} \frac{\Gamma\{P, \langle P \rangle\}}{\Gamma\{P\}} & \square \frac{\Gamma\{[A]\}}{\Gamma\{\square A\}} & \bar{\vee} \frac{\Gamma\{A, B\}}{\Gamma\{A \bar{\vee} B\}} & \bar{\wedge} \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \bar{\wedge} B\}} & \text{sto} \frac{\Gamma\{P\}}{\Gamma\{\uparrow P\}} \\ \text{id} \frac{}{\Gamma\{\bar{a}, \langle a \rangle\}} & k^\circ \frac{\Gamma\{\langle A \rangle, \Delta\}}{\Gamma\{\langle \diamond A \rangle, [\Delta]\}} & \hat{\wedge} \frac{\Gamma\{\langle A \rangle\} \quad \Gamma\{\langle B \rangle\}}{\Gamma\{\langle A \hat{\wedge} B \rangle\}} & \dot{\vee}_i \frac{\Gamma\{\langle A_i \rangle\}}{\Gamma\{\langle A_1 \dot{\vee} A_2 \rangle\}} & \text{rel} \frac{\Gamma\{N\}}{\Gamma\{\langle \downarrow N \rangle\}} \end{array}$$

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$$\begin{array}{ccccc} d^\circ \frac{\Gamma\{\langle A \rangle\}}{\Gamma\{\langle \diamond A \rangle\}} & t^\circ \frac{\Gamma\{\langle A \rangle\}}{\Gamma\{\langle \diamond A \rangle\}} & b^\circ \frac{\Gamma\{[\Delta], \langle A \rangle\}}{\Gamma\{[\Delta], \langle \diamond A \rangle\}} & 4^\circ \frac{\Gamma\{\langle \diamond A \rangle, \Delta\}}{\Gamma\{\langle \diamond A \rangle, [\Delta]\}} & 5^\circ \frac{\Gamma\{\emptyset\}\{\langle \diamond A \rangle\}}{\Gamma\{\langle \diamond A \rangle\}\{\emptyset\}} \end{array}$$



# The synthetic nested system

## Synthetic system KNS:

$$\text{dec} \frac{\Gamma\{P, \langle P \rangle\}}{\Gamma\{P\}}$$

$$\text{neg} \frac{\{\Gamma\{\Delta\}\}_{\Delta \leq N}}{\Gamma\{N\}}$$

$$\text{id} \frac{}{\Gamma\{\bar{a}, \langle a \rangle\}}$$

$$k^\diamond \frac{\Gamma\{\langle A \rangle, \Delta\}}{\Gamma\{\langle \diamond A \rangle, [\Delta]\}}$$

$$\hat{\wedge} \frac{\Gamma\{\langle A \rangle\} \quad \Gamma\{\langle B \rangle\}}{\Gamma\{\langle A \hat{\wedge} B \rangle\}}$$

$$\hat{\vee}_i \frac{\Gamma\{\langle A_i \rangle\}}{\Gamma\{\langle A_1 \hat{\vee} A_2 \rangle\}}$$

$$\text{rel} \frac{\Gamma\{N\}}{\Gamma\{\langle \downarrow N \rangle\}}$$

## Focused modal rules:

$$d^\diamond \frac{\Gamma\{\langle A \rangle\}}{\Gamma\{\langle \diamond A \rangle\}}$$

$$t^\diamond \frac{\Gamma\{\langle A \rangle\}}{\Gamma\{\langle \diamond A \rangle\}}$$

$$b^\diamond \frac{\Gamma\{[\Delta], \langle A \rangle\}}{\Gamma\{[\Delta], \langle \diamond A \rangle\}}$$

$$4^\diamond \frac{\Gamma\{\langle \diamond A \rangle, \Delta\}}{\Gamma\{\langle \diamond A \rangle, [\Delta]\}}$$

$$5^\diamond \frac{\Gamma\{\emptyset\}\{\langle \diamond A \rangle\}}{\Gamma\{\langle \diamond A \rangle\}\{\emptyset\}}$$

# The synthetic nested system

## Synthetic system KNS:

$$\text{pos} \frac{\Delta \preceq \bar{P} \quad \Gamma\{P, \langle \Delta \rangle\}}{\Gamma\{P\}}$$

$$\text{neg} \frac{\{\Gamma\{\Delta\}\}_{\Delta \preceq N}}{\Gamma\{N\}}$$

$$\text{id} \frac{}{\Gamma\{\bar{a}, \langle a \rangle\}}$$

$$k^\diamond \frac{\Gamma\{\langle A \rangle, \Delta\}}{\Gamma\{\langle \diamond A \rangle, [\Delta]\}}$$

$$\text{split} \frac{\Gamma\{\langle \Delta_1 \rangle\} \quad \Gamma\{\langle \Delta_2 \rangle\}}{\Gamma\{\langle \Delta_1, \Delta_2 \rangle\}}$$

$$\text{rel} \frac{\Gamma\{N\}}{\Gamma\{\langle \downarrow N \rangle\}}$$

## Focused modal rules:

$$d^\diamond \frac{\Gamma\{\langle A \rangle\}}{\Gamma\{\langle \diamond A \rangle\}}$$

$$t^\diamond \frac{\Gamma\{\langle A \rangle\}}{\Gamma\{\langle \diamond A \rangle\}}$$

$$b^\diamond \frac{\Gamma\{[\Delta], \langle A \rangle\}}{\Gamma\{[\Delta], \langle \diamond A \rangle\}}$$

$$4^\diamond \frac{\Gamma\{\langle \diamond A \rangle, \Delta\}}{\Gamma\{\langle \diamond A \rangle, [\Delta]\}}$$

$$5^\diamond \frac{\Gamma\{\emptyset\}\{\langle \diamond A \rangle\}}{\Gamma\{\langle \diamond A \rangle\}\{\emptyset\}}$$

# The synthetic nested system

## Synthetic substructure matching:

$$\begin{array}{c} \Downarrow \bar{\vee} \\ \frac{\Gamma \Downarrow M \quad \Delta \Downarrow N}{\Gamma, \Delta \Downarrow M \bar{\vee} N} \end{array} \quad \Downarrow \bar{\wedge}_i \frac{\Gamma \Downarrow N_i}{\Gamma \Downarrow N_1 \bar{\wedge} N_2} \quad \Downarrow \square \frac{\Gamma \Downarrow N}{[\Gamma] \Downarrow \square N} \quad \Downarrow \uparrow \frac{}{P \Downarrow \uparrow P} \quad \Downarrow \text{id} \frac{}{\bar{a} \Downarrow \bar{a}}$$

## Synthetic system KNS:

$$\text{pos} \frac{\Delta \Downarrow \bar{P} \quad \Gamma\{P, \langle \Delta \rangle\}}{\Gamma\{P\}} \quad \text{neg} \frac{\{\Gamma\{\Delta\}\}_{\Delta \Downarrow N}}{\Gamma\{N\}}$$

$$\text{id} \frac{}{\Gamma\{\bar{a}, \langle a \rangle\}} \quad \text{k}^\diamond \frac{\Gamma\{\langle A \rangle, \Delta\}}{\Gamma\{\langle \diamond A \rangle, [\Delta]\}} \quad \text{split} \frac{\Gamma\{\langle \Delta_1 \rangle\} \quad \Gamma\{\langle \Delta_2 \rangle\}}{\Gamma\{\langle \Delta_1, \Delta_2 \rangle\}} \quad \text{rel} \frac{\Gamma\{N\}}{\Gamma\{\langle \downarrow N \rangle\}}$$

## Focused modal rules:

$$\text{d}^\diamond \frac{\Gamma\{\langle A \rangle\}}{\Gamma\{\langle \diamond A \rangle\}} \quad \text{t}^\diamond \frac{\Gamma\{\langle A \rangle\}}{\Gamma\{\langle \diamond A \rangle\}} \quad \text{b}^\diamond \frac{\Gamma\{[\Delta], \langle A \rangle\}}{\Gamma\{[\Delta], \langle \diamond A \rangle\}} \quad \text{4}^\diamond \frac{\Gamma\{\langle \diamond A \rangle, \Delta\}}{\Gamma\{\langle \diamond A \rangle, [\Delta]\}} \quad \text{5}^\diamond \frac{\Gamma\{\emptyset\}\{\langle \diamond A \rangle\}}{\Gamma\{\langle \diamond A \rangle\}\{\emptyset\}}$$

# The synthetic nested system

## Synthetic substructure matching:

$$\begin{array}{c} \Downarrow \checkmark \\ \frac{\Gamma \Downarrow M \quad \Delta \Downarrow N}{\Gamma, \Delta \Downarrow M \checkmark N} \end{array} \quad \Downarrow \bar{\wedge}_i \frac{\Gamma \Downarrow N_i}{\Gamma \Downarrow N_1 \bar{\wedge} N_2} \quad \Downarrow \square \frac{\Gamma \Downarrow N}{[\Gamma] \Downarrow \square N} \quad \Downarrow \uparrow \frac{}{P \Downarrow \uparrow P} \quad \Downarrow \text{id} \frac{}{\bar{a} \Downarrow \bar{a}}$$

## Synthetic system KNS:

$$\begin{array}{c} \text{pos} \frac{\Delta \Downarrow \bar{P} \quad \Gamma\{P, \langle \Delta \rangle\}}{\Gamma\{P\}} \end{array} \quad \begin{array}{c} \text{neg} \frac{\{\Gamma\{\Delta\}\}_{\Delta \Downarrow N}}{\Gamma\{N\}} \end{array}$$

$$\begin{array}{c} \text{id} \frac{}{\Gamma\{\bar{a}, \langle \bar{a} \rangle\}} \end{array} \quad \begin{array}{c} \text{k}^\diamond \frac{\Gamma\{\langle A \rangle, \Delta\}}{\Gamma\{\langle \diamond A \rangle, [\Delta]\}} \end{array} \quad \begin{array}{c} \text{split} \frac{\Gamma\{\langle \Delta_1 \rangle\} \quad \Gamma\{\langle \Delta_2 \rangle\}}{\Gamma\{\langle \Delta_1, \Delta_2 \rangle\}} \end{array} \quad \begin{array}{c} \text{rel} \frac{\Gamma\{\bar{P}\}}{\Gamma\{\langle P \rangle\}} \end{array}$$

## Focused modal rules:

$$\begin{array}{c} \text{d}^\diamond \frac{\Gamma\{\langle A \rangle\}}{\Gamma\{\langle \diamond A \rangle\}} \end{array} \quad \begin{array}{c} \text{t}^\diamond \frac{\Gamma\{\langle A \rangle\}}{\Gamma\{\langle \diamond A \rangle\}} \end{array} \quad \begin{array}{c} \text{b}^\diamond \frac{\Gamma\{\langle \Delta \rangle, \langle A \rangle\}}{\Gamma\{\langle \Delta, \langle \diamond A \rangle \}} \end{array} \quad \begin{array}{c} \text{4}^\diamond \frac{\Gamma\{\langle \langle \diamond A \rangle, \Delta \}}{\Gamma\{\langle \diamond A \rangle, [\Delta]\}} \end{array} \quad \begin{array}{c} \text{5}^\diamond \frac{\Gamma\{\emptyset\}\{\langle \diamond A \rangle\}}{\Gamma\{\langle \diamond A \rangle\}\{\emptyset\}} \end{array}$$

# The synthetic nested system

## Synthetic substructure matching:

$$\begin{array}{c} \Downarrow \bar{\vee} \\ \frac{\Gamma \Downarrow M \quad \Delta \Downarrow N}{\Gamma, \Delta \Downarrow M \bar{\vee} N} \end{array} \quad \Downarrow \bar{\wedge}_i \frac{\Gamma \Downarrow N_i}{\Gamma \Downarrow N_1 \bar{\wedge} N_2} \quad \Downarrow \square \frac{\Gamma \Downarrow N}{[\Gamma] \Downarrow \square N} \quad \Downarrow \uparrow \frac{}{P \Downarrow \uparrow P} \quad \Downarrow \text{id} \frac{}{\bar{a} \Downarrow \bar{a}}$$

## Synthetic system KNS:

$$\begin{array}{c} \text{pos} \frac{\Delta \Downarrow \bar{P} \quad \Gamma\{P, \langle \Delta \rangle\}}{\Gamma\{P\}} \quad \text{neg} \frac{\{\Gamma\{\Delta\}\}_{\Delta \Downarrow N}}{\Gamma\{N\}} \\ \text{id} \frac{}{\Gamma\{\bar{a}, \langle \bar{a} \rangle\}} \quad \text{k} \frac{\Gamma\{[\langle \Delta \rangle, \Omega]\}}{\Gamma\{[\langle \Delta \rangle, [\Omega]]\}} \quad \text{split} \frac{\Gamma\{\langle \Delta_1 \rangle\} \quad \Gamma\{\langle \Delta_2 \rangle\}}{\Gamma\{\langle \Delta_1, \Delta_2 \rangle\}} \quad \text{rel} \frac{\Gamma\{\bar{P}\}}{\Gamma\{\langle P \rangle\}} \end{array}$$

## Focused modal rules:

$$\begin{array}{c} \text{d} \frac{\Gamma\{[\langle A \rangle]\}}{\Gamma\{\langle \diamond A \rangle\}} \quad \text{t} \frac{\Gamma\{\langle A \rangle\}}{\Gamma\{\langle \diamond A \rangle\}} \quad \text{b} \frac{\Gamma\{[\langle \Delta \rangle, \langle A \rangle]\}}{\Gamma\{[\langle \Delta \rangle, \langle \diamond A \rangle]\}} \quad \text{4} \frac{\Gamma\{[\langle \diamond A \rangle, \Delta]\}}{\Gamma\{\langle \diamond A \rangle, [\Delta]\}} \quad \text{5} \frac{\Gamma\{\emptyset\}\{\langle \diamond A \rangle\}}{\Gamma\{\langle \diamond A \rangle\}\{\emptyset\}} \end{array}$$

# The synthetic nested system

## Synthetic substructure matching:

$$\begin{array}{c} \Downarrow \\ \frac{\Gamma \Downarrow M \quad \Delta \Downarrow N}{\Gamma, \Delta \Downarrow M \Downarrow N} \end{array} \quad \Downarrow \bar{\wedge}_i \frac{\Gamma \Downarrow N_i}{\Gamma \Downarrow N_1 \bar{\wedge} N_2} \quad \Downarrow \square \frac{\Gamma \Downarrow N}{[\Gamma] \Downarrow \square N} \quad \Downarrow \uparrow \frac{}{P \Downarrow \uparrow P} \quad \Downarrow \text{id} \frac{}{\bar{a} \Downarrow \bar{a}}$$

## Synthetic system KNS:

$$\begin{array}{c} \text{pos} \frac{\Delta \Downarrow \bar{P} \quad \Gamma\{P, \langle \Delta \rangle\}}{\Gamma\{P\}} \quad \text{neg} \frac{\{\Gamma\{\Delta\}\}_{\Delta \Downarrow N}}{\Gamma\{N\}} \\ \text{id} \frac{}{\Gamma\{\bar{a}, \langle \bar{a} \rangle\}} \quad \text{k}^\diamond \frac{\Gamma\{\langle \Delta \rangle, \Omega\}}{\Gamma\{\langle \Delta \rangle, [\Omega]\}} \quad \text{split} \frac{\Gamma\{\langle \Delta_1 \rangle\} \quad \Gamma\{\langle \Delta_2 \rangle\}}{\Gamma\{\langle \Delta_1, \Delta_2 \rangle\}} \quad \text{rel} \frac{\Gamma\{\bar{P}\}}{\Gamma\{\langle P \rangle\}} \end{array}$$

## Synthetic modal rules:

$$\begin{array}{c} \text{d}^\diamond \frac{\Gamma\{\langle \langle \Delta \rangle \rangle\}}{\Gamma\{\langle [\Delta] \rangle\}} \quad \text{t}^\diamond \frac{\Gamma\{\langle \Delta \rangle\}}{\Gamma\{\langle [\Delta] \rangle\}} \quad \text{b}^\diamond \frac{\Gamma\{[\Omega], \langle \Delta \rangle\}}{\Gamma\{[\Omega], \langle [\Delta] \rangle\}} \quad \text{4}^\diamond \frac{\Gamma\{[\Omega], \langle [\Delta] \rangle\}}{\Gamma\{[\Omega], \langle [\Delta] \rangle\}} \quad \text{5}^\diamond \frac{\Gamma\{\langle [\Delta] \rangle\} \{\emptyset\}}{\Gamma\{\emptyset\} \langle [\Delta] \rangle} \end{array}$$

# In action...

**Synthetic connectives:**  $\text{neg} \frac{\{\Gamma\{\Delta\}\}_{\Delta \preceq N}}{\Gamma\{N\}}$  and  $\text{pos} \frac{\Delta \preceq \bar{P} \quad \Gamma\{P, \langle \Delta \rangle\}}{\Gamma\{P\}}$

**Structural modal rules :** distinct modal phase and action on substructures

$$\begin{array}{c}
 \text{id} \\
 \text{K}^{\diamond} \frac{\overline{\diamond(a \dot{\vee} b), [\langle \bar{a} \rangle, \bar{a}]}}{\diamond(a \dot{\vee} b), \langle \bar{a} \rangle, [\bar{a}]} \\
 \text{pos} \frac{\overline{\diamond(a \dot{\vee} b), [\bar{a}]}}{\diamond(a \dot{\vee} b), [\bar{a}]} \\
 \text{neg} \frac{\overline{\diamond(a \dot{\vee} b), [\bar{a}]}}{\diamond(a \dot{\vee} b), \square(\bar{a} \bar{\wedge} \bar{b})}
 \end{array}
 \qquad
 \begin{array}{c}
 \text{id} \\
 \text{K}^{\diamond} \frac{\overline{\diamond(a \dot{\vee} b), [\langle \bar{b} \rangle, \bar{b}]}}{\diamond(a \dot{\vee} b), \langle \bar{b} \rangle, [\bar{b}]} \\
 \text{pos} \frac{\overline{\diamond(a \dot{\vee} b), [\bar{b}]}}{\diamond(a \dot{\vee} b), [\bar{b}]}
 \end{array}$$

$$\begin{array}{c}
 \preceq \text{id} \frac{\overline{\bar{a} \preceq \bar{a}}}{\bar{a} \preceq \bar{a}} \\
 \preceq \bar{\wedge} \frac{\overline{\bar{a} \preceq \bar{a} \bar{\wedge} \bar{b}}}{\bar{a} \preceq \bar{a} \bar{\wedge} \bar{b}} \\
 \preceq \square \frac{\overline{[\bar{a}] \preceq \square(\bar{a} \bar{\wedge} \bar{b})}}{[\bar{a}] \preceq \square(\bar{a} \bar{\wedge} \bar{b})}
 \end{array}$$

$$\begin{array}{c}
 \preceq \text{id} \frac{\overline{\bar{b} \preceq \bar{a}}}{\bar{b} \preceq \bar{a}} \\
 \preceq \bar{\wedge} \frac{\overline{\bar{b} \preceq \bar{a} \bar{\wedge} \bar{b}}}{\bar{b} \preceq \bar{a} \bar{\wedge} \bar{b}} \\
 \preceq \square \frac{\overline{[\bar{b}] \preceq \square(\bar{a} \bar{\wedge} \bar{b})}}{[\bar{b}] \preceq \square(\bar{a} \bar{\wedge} \bar{b})}
 \end{array}$$

# Synthetic permutation

$$\text{cut} \frac{\text{rel} \frac{\mathcal{D}_1}{\Gamma\{N\}\{P\}} \quad \Gamma\{\langle \downarrow N \rangle\}\{P\}}{\Gamma\{\langle \downarrow N \rangle\}\{\emptyset\}} \quad \Gamma\{\emptyset\}\{\bar{P}\}}{\Gamma\{\langle \downarrow N \rangle\}\{\emptyset\}} \quad \mathcal{D}_2$$

$$\sim$$

$$\text{cut} \frac{\mathcal{D}_1 \quad \Gamma\{N\}\{P\}}{\text{rel} \frac{\Gamma\{N\}\{\emptyset\}}{\Gamma\{\langle \downarrow N \rangle\}\{\emptyset\}}} \quad \text{weak} \frac{\Gamma\{\emptyset\}\{\bar{P}\}}{\Gamma\{N\}\{\bar{P}\}}$$



# Synthetic permutation

$$\begin{array}{c}
 \left\{ \begin{array}{c} \triangle_{D_\Delta} \\ \Gamma\{\Delta\}\{P\} \end{array} \right\} \Delta \preceq \bar{Q} \\
 \text{neg} \frac{}{\Gamma\{\bar{Q}\}\{P\}} \\
 \text{rel} \frac{}{\Gamma\{\langle Q \rangle\}\{P\}} \\
 \text{cut} \frac{}{\Gamma\{\langle Q \rangle\}\{\emptyset\}} \quad \triangle_{D_2} \quad \Gamma\{\emptyset\}\{\bar{P}\} \\
 \hline
 \end{array}
 \quad \rightsquigarrow \quad
 \begin{array}{c}
 \left\{ \begin{array}{c} \triangle_{D_\Delta} \quad \triangle_{D_2} \\ \Gamma\{\Delta\}\{P\} \quad \text{weak} \frac{\Gamma\{\emptyset\}\{\bar{P}\}}{\Gamma\{\Delta\}\{\bar{P}\}} \\ \text{cut} \frac{}{\Gamma\{\Delta\}\{\emptyset\}} \end{array} \right\} \Delta \preceq \bar{Q} \\
 \text{neg} \frac{}{\Gamma\{\bar{Q}\}\{\emptyset\}} \\
 \text{rel} \frac{}{\Gamma\{\langle Q \rangle\}\{\emptyset\}} \\
 \hline
 \end{array}$$

# Synthetic permutation

$$\text{neg} \frac{\left\{ \begin{array}{c} \mathcal{D}_\Delta \\ \Gamma\{\Delta\}\{P\} \end{array} \right\} \Delta \preceq \bar{Q}}{\Gamma\{\bar{Q}\}\{P\}} \quad \mathcal{D}_2 \quad \Gamma\{\emptyset\}\{\bar{P}\} \\
 \text{rel} \frac{\Gamma\{\langle Q \rangle\}\{P\}}{\Gamma\{\langle Q \rangle\}\{\emptyset\}} \\
 \text{cut} \frac{\Gamma\{\bar{Q}\}\{P\}}{\Gamma\{\langle Q \rangle\}\{\emptyset\}} \quad \sim \quad \text{neg} \frac{\left\{ \begin{array}{c} \mathcal{D}_\Delta \quad \mathcal{D}_2 \\ \text{cut} \frac{\Gamma\{\Delta\}\{P\} \quad \text{weak} \frac{\Gamma\{\emptyset\}\{\bar{P}\}}{\Gamma\{\Delta\}\{\bar{P}\}}}{\Gamma\{\Delta\}\{\emptyset\}} \end{array} \right\} \Delta \preceq \bar{Q}}{\text{rel} \frac{\Gamma\{\bar{Q}\}\{\emptyset\}}{\Gamma\{\langle Q \rangle\}\{\emptyset\}}}$$

KN  $\longrightarrow$  KNS + cut  $\longrightarrow$  KNS



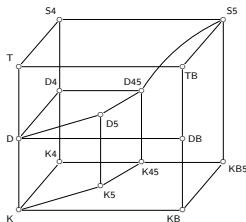
# Cut-elimination

**Theorem** Let  $X \subseteq \{d, t, b, 4, 5\}$  be 45-closed.

If a sequent  $\Gamma$  is provable in  $\text{KNF} + X^\diamond + \text{Cut}$ , then it is also provable in  $\text{KNF} + X^\diamond$ .

$$\text{Cut} = \left\{ \text{cut}_1 \frac{\Gamma\{P\} \quad \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}}; \text{cut}_2 \frac{\Gamma\{\langle P \rangle\} \quad \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}}; \text{cut}_3 \frac{\Gamma\{\langle Q \rangle\}\{P\} \quad \Gamma\{\emptyset\}\{\bar{P}\}}{\Gamma\{\langle Q \rangle\}\{\emptyset\}} \right\}$$

$$\text{clo}(X) = \begin{cases} X \cup \{4\} & \text{if } \{b, 5\} \subseteq X \text{ or if } \{t, 5\} \subseteq X \\ X \cup \{5\} & \text{if } \{b, 4\} \subseteq X \\ X & \text{otherwise} \end{cases}$$



# Cut-elimination proof

$$\begin{array}{c}
 \text{id} \frac{}{[\bar{a}, a], [\diamond a]} \\
 \diamond \\
 \square \frac{[\bar{a}], \diamond a, [\diamond a]}{\square \bar{a}, \diamond a, [\diamond a]} \\
 \text{b}^\diamond \\
 \text{cut} \frac{\square \bar{a}, [\diamond \diamond a, \diamond a]}{\square \bar{a}, [\diamond a]}
 \end{array}
 \quad
 \begin{array}{c}
 \text{id} \frac{}{\square \bar{a}, [[[\bar{a}, a]]]} \\
 \diamond \\
 \square \frac{\square \bar{a}, [[[\bar{a}], \diamond a]]}{\square \bar{a}, [[\square \bar{a}, \diamond a]]} \\
 4^\diamond \\
 \square \frac{\square \bar{a}, [[\square \bar{a}], \diamond a]}{\square \bar{a}, [\square \square \bar{a}, \diamond a]}
 \end{array}
 \quad
 \rightsquigarrow
 \quad
 \begin{array}{c}
 \text{id} \frac{}{[\bar{a}, a], [ ]} \\
 \diamond \\
 \square \frac{[\bar{a}], \diamond a, [ ]}{\square \bar{a}, \diamond a, [ ]} \\
 5^\diamond \\
 \square \bar{a}, [\diamond a]
 \end{array}$$

# Completeness proof

KN

$$\begin{array}{l} \text{id} \frac{}{\overline{[\bar{a}, a], [ ]}} \\ \diamond \frac{}{\overline{[\bar{a}], \diamond a, [ ]}} \\ \square \frac{}{\overline{\square \bar{a}, \diamond a, [ ]}} \\ 5^\circ \frac{}{\overline{\square \bar{a}, [\diamond a]}} \\ \square \frac{}{\overline{\square \bar{a}, \square \diamond a}} \end{array}$$

# Completeness proof

simulation

KN  $\longrightarrow$  KNwF + cut

$$\begin{array}{l} \text{id} \frac{}{\overline{[\bar{a}, a], [ ]}} \\ \diamond \frac{}{\overline{[\bar{a}], \diamond a, [ ]}} \\ \square \frac{}{\overline{\square \bar{a}, \diamond a, [ ]}} \\ 5^\circ \frac{}{\overline{\square \bar{a}, [\diamond a]}} \\ \square \frac{}{\overline{\square \bar{a}, \square \diamond a}} \end{array}$$

(1)

# Completeness proof

KN  $\longrightarrow$  KNwF + cut  $\xrightarrow{\text{cut-elimination}}$  KNwF



# Completeness proof

rules permutation

KN  $\longrightarrow$  KNwF + cut  $\longrightarrow$  KNwF  $\longrightarrow$  KNF

$$\begin{array}{c} \text{id} \frac{}{[\bar{a}, \langle a \rangle], [\diamond a]} \\ \diamond \\ \square \frac{}{[\bar{a}], \langle \diamond a \rangle, [\diamond a]} \\ 5^\circ \frac{}{\square \bar{a}, \langle \diamond a \rangle, [\diamond a]} \\ \text{dec} \frac{}{\square \bar{a}, [\diamond a, \langle \diamond a \rangle]} \\ \text{sto} \frac{}{\square \bar{a}, [\diamond a]} \\ \square \frac{}{\square \bar{a}, [\uparrow \diamond a]} \\ \square \frac{}{\square \bar{a}, \square \uparrow \diamond a} \end{array}$$

$$\begin{array}{c} \text{id} \frac{}{[\bar{a}, \langle a \rangle], [\diamond a]} \\ \diamond \\ 5^\circ \frac{}{[\bar{a}], \langle \diamond a \rangle, [\diamond a]} \\ \text{dec} \frac{}{[\bar{a}], [\diamond a, \langle \diamond a \rangle]} \\ \text{sto} \frac{}{[\bar{a}], [\diamond a]} \\ \square \frac{}{[\bar{a}], [\uparrow \diamond a]} \\ \square \frac{}{\square \bar{a}, [\uparrow \diamond a]} \\ \square \frac{}{\square \bar{a}, \square \uparrow \diamond a} \end{array}$$

# In action...

- **neg** and **pos** : synthetic connectives
- structural modal rules : modal phase / action on substructures

$$\begin{array}{c}
 \text{id}^0 \frac{}{\diamond \downarrow (\bar{a} \bar{\wedge} \uparrow b), \diamond a, [\langle \bar{a} \rangle, \bar{a}, \bar{b}]} \\
 \text{k}^0 \frac{}{\diamond \downarrow (\bar{a} \bar{\wedge} \uparrow b), \diamond a, \langle [\bar{a}] \rangle, [\bar{a}, \bar{b}]} \\
 \text{pos}^0 \frac{}{\diamond \downarrow (\bar{a} \bar{\wedge} \uparrow b), \diamond a, [\bar{a}, \bar{b}]} \\
 \text{neg}^0 \frac{}{\diamond \downarrow (\bar{a} \bar{\wedge} \uparrow b), \diamond a, [\bar{a}, \bar{b}]} \\
 \text{rel}^0 \frac{}{\diamond \downarrow (\bar{a} \bar{\wedge} \uparrow b), \diamond a, [\bar{a} \bar{\wedge} \uparrow b, \bar{b}]} \\
 \text{k}^0 \frac{}{\diamond \downarrow (\bar{a} \bar{\wedge} \uparrow b), \diamond a, [\langle a \downarrow \bar{b} \rangle, \bar{b}]} \\
 \text{pos}^0 \frac{}{\diamond \downarrow (\bar{a} \bar{\wedge} \uparrow b), \langle [a \downarrow \bar{b}] \rangle, \diamond a, [\bar{b}]} \\
 \text{pos}^0 \frac{}{\diamond \downarrow (\bar{a} \bar{\wedge} \uparrow b), \diamond a, [\bar{b}]}
 \end{array}$$

$$\begin{array}{c}
 \text{id}^0 \frac{}{\diamond (a \bar{\wedge} b), \diamond \downarrow \bar{a}, [\langle \bar{a} \rangle, \bar{a}, \bar{b}]} \quad \text{id}^0 \frac{}{\diamond (a \bar{\wedge} b), \diamond \downarrow \bar{a}, [\langle \bar{b} \rangle, \bar{a}, \bar{b}]} \\
 \text{split}^0 \frac{}{\diamond (a \bar{\wedge} b), \diamond \downarrow \bar{a}, [\langle \bar{a} \rangle, \bar{a}, \bar{b}]} \\
 \text{k}^0 \frac{}{\diamond (a \bar{\wedge} b), \langle [\bar{a}, \bar{b}] \rangle, \diamond \downarrow \bar{a}, [\bar{a}, \bar{b}]} \\
 \text{pos}^0 \frac{}{\diamond (a \bar{\wedge} b), \diamond \downarrow \bar{a}, [\bar{a}, \bar{b}]} \\
 \text{rel}^0 \frac{}{\diamond (a \bar{\wedge} b), \diamond \downarrow \bar{a}, [\langle a \rangle, \bar{b}]} \\
 \text{k}^0 \frac{}{\diamond (a \bar{\wedge} b), \diamond \downarrow \bar{a}, \langle [a] \rangle, [\bar{b}]} \\
 \text{pos}^0 \frac{}{\diamond (a \bar{\wedge} b), \diamond \downarrow \bar{a}, [\bar{b}]}
 \end{array}$$