Label-free Modular Systems for Classical and Intuitionistic Modal Logics

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Classical Modal Logic

- Formulas:

$$A, B, \ldots ::= p \mid \bar{p} \mid A \wedge B \mid A \vee B \mid \Box A \mid \Diamond A$$

- Negation: De Morgan laws and $\overline{\Box A} = \Diamond \bar{A}$

- Axioms for K: classical propositional logic and

$$\mathsf{k}: \quad \Box(A \supset B) \supset (\Box A \supset \Box B)$$

- Rules: modus ponens: $\dfrac{A \quad A \supset B}{B}$ necessitation: $\dfrac{A}{\Box A}$

Intuitionistic Modal Logic

- Formulas:

$$A, B, \ldots ::= p \mid \bot \mid A \wedge B \mid A \vee B \mid A \supset B \mid \Box A \mid \Diamond A$$

- Negation: $\neg A = A \supset \bot$ and independance of the modalities

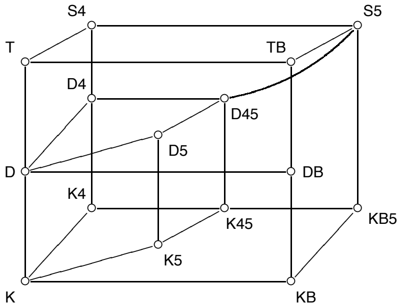
- Axioms for IK: intuitionistic propositional logic and

$$\begin{array}{ll}
k_1: & \Box(A \supset B) \supset (\Box A \supset \Box B) \\
k_2: & \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) \\
k_3: & \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B) \\
k_4: & (\Diamond A \supset \Box B) \supset \Box(A \supset B) \\
k_5: & \neg\Diamond\bot
\end{array}$$

- Rules: modus ponens: $\dfrac{A \quad A \supset B}{B}$ necessitation: $\dfrac{A}{\Box A}$

Classical Modal Axioms

d: $\Box A \supset \Diamond A$
t: $A \supset \Diamond A$
b: $A \supset \Box\Diamond A$
4: $\Diamond\Diamond A \supset \Diamond A$
5: $\Diamond A \supset \Box\Diamond A$



Intuitionistic Modal Axioms

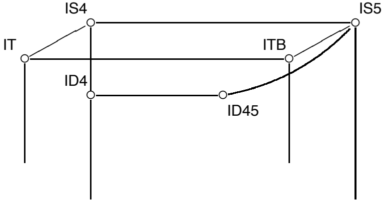
d: $\Box A \supset \Diamond A$

t: $A \supset \Diamond A \quad \wedge \quad \Box A \supset A$

b: $A \supset \Box \Diamond A \quad \wedge \quad \Diamond \Box A \supset A$

4: $\Diamond \Diamond A \supset \Diamond A \quad \wedge \quad \Box A \supset \Box \Box A$

5: $\Diamond A \supset \Box \Diamond A \quad \wedge \quad \Diamond \Box A \supset \Box A$



Nested Sequents for classical modal logic

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- Sequent:

$$\Gamma ::= A_1, \ldots, A_m$$

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- Corresponding formula:

$$fm(\Gamma) = A_1 \vee \ldots \vee A_m$$

Nested Sequents for classical modal logic

- Nested Sequent:

$$\Gamma ::= A_1, \ldots, A_m, [\Gamma_1], \ldots, [\Gamma_n]$$

- Corresponding formula:

$$fm(\Gamma) = A_1 \vee \ldots \vee A_m \vee \Box fm(\Gamma_1) \vee \ldots \vee \Box fm(\Gamma_n)$$

Nested Sequents for classical modal logic

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$$\Gamma ::= A_1, \ldots, A_m, [\Gamma_1], \ldots, [\Gamma_n]$$

- Corresponding formula:

$$fm(\Gamma) = A_1 \vee \ldots \vee A_m \vee \Box fm(\Gamma_1) \vee \ldots \vee \Box fm(\Gamma_n)$$

- A context is a sequent with one or several holes:

$$\Gamma\{\ \}\{\ \} = A, [B, \{\ \}, [\{\ \}], C]$$

Nested Sequents for classical modal logic

- Nested Sequent:

$$\Gamma ::= A_1, \ldots, A_m, [\Gamma_1], \ldots, [\Gamma_n]$$

- Corresponding formula:

$$fm(\Gamma) = A_1 \vee \ldots \vee A_m \vee \Box fm(\Gamma_1) \vee \ldots \vee \Box fm(\Gamma_n)$$

- A context is a sequent with one or several holes:

$$\Gamma\{\ \}\{\ \} = A, [B, \{\ \}, [\{\ \}], C]$$

$$\Gamma\{[D]\}\{A, [C]\} = A, [B, [D], [A, [C]], C]$$

Nested Sequents for intuitionistic modal logic

- Sequent:

$$\Gamma ::= A_1, \ldots, A_m \vdash B$$

- Corresponding formula:

$$A_1 \wedge \ldots \wedge A_m \supset B$$

Nested Sequents for intuitionistic modal logic

- Sequent:

$$\Gamma ::= A_1, \ldots, A_m, B$$

- Corresponding formula:

$$A_1 \wedge \ldots \wedge A_m \supset B$$

Nested Sequents for intuitionistic modal logic

- Nested Sequent:

$$\Gamma ::= \Lambda \ , \Pi$$

- Corresponding formula:

$$fm(\Gamma) = fm(\Lambda \ ) \supset fm(\Pi \ )$$

Nested Sequents for intuitionistic modal logic

- Nested Sequent:

$$\Gamma ::= \Lambda \;, \Pi$$

$$\Lambda \;\; ::= A_1, ..., A_m, [\Lambda_1], ..., [\Lambda_n]$$

$$\Pi \;\; ::= A \;\; \mid [\Gamma]$$

- Corresponding formula:

$$fm(\Gamma) = fm(\Lambda\;) \supset fm(\Pi\;)$$

$$fm(\Lambda\;) = A_1 \wedge ... \wedge A_m \wedge \Diamond fm(\Lambda_1) \wedge ... \wedge \Diamond fm(\Lambda_n)$$

$$fm([\Gamma]) = \Box fm(\Gamma)$$

Nested Sequent for intuitionistic modal logic

- Output context:

$$\Gamma_1\{\ \} = A\ \ , [B\ \ , \{\ \}]$$

Nested Sequent for intuitionistic modal logic

- Output context:

$$\Gamma_1\{\ \} = A\ , [B\ , \{\ \}]$$

$$\to \Gamma_1\{[C\ , D\ ]\} = A\ , [B\ , [C\ , D\ ]]$$

Nested Sequent for intuitionistic modal logic

- Output context:

$$\Gamma_1\{\ \} = A\ , [B\ , \{\ \}]$$

$$\to \Gamma_1\{[C\ , D\ ]\} = A\ , [B\ , [C\ , D\ ]]$$

- Input context

$$\Gamma_2\{\ \} = A\ , [B\ , \{\ \}]$$

Nested Sequent for intuitionistic modal logic

- Output context:

$$\Gamma_1\{\ \} = A\ ,[B\ ,\{\ \}]$$

$$\to \Gamma_1\{[C\ ,D\ ]\} = A\ ,[B\ ,[C\ ,D\ ]]$$

- Input context

$$\Gamma_2\{\ \} = A\ ,[B\ ,\{\ \}]$$

$$\to \Gamma_2\{[C\ ,D\ ]\} = A\ ,[B\ ,[C\ ,D\ ]]$$

Classical Rules

System NK

$$\mathsf{id}\ \frac{}{\Gamma\{a, \bar{a}\}} \qquad \vee\ \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}} \qquad \wedge\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}}$$

$$\mathsf{c}\ \frac{\Gamma\{A, A\}}{\Gamma\{A\}} \qquad \Box\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \Diamond\ \frac{\Gamma\{[A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}$$

Additional structural rules

$$\mathsf{w}\ \frac{\Gamma\{\emptyset\}}{\Gamma\{\Delta\}} \qquad \mathsf{cut}\ \frac{\Gamma\{\bar{A}\} \quad \Gamma\{A\}}{\Gamma\{\emptyset\}}$$

Classical Rules

Modal $\Diamond$-rules

$$\mathsf{d}^{\Diamond}\ \frac{\Gamma\{[A]\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{t}^{\Diamond}\ \frac{\Gamma\{A\}}{\Gamma\{\Diamond A\}} \qquad \mathsf{b}^{\Diamond}\ \frac{\Gamma\{[\Delta], A\}}{\Gamma\{[\Delta, \Diamond A]\}}$$

$$4^{\Diamond}\ \frac{\Gamma\{[\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} \qquad 5^{\Diamond}\ \frac{\Gamma\{\emptyset\}\{\Diamond A\}}{\Gamma\{\Diamond A\}\{\emptyset\}}\ \mathit{depth}(\Gamma\{\ \}\{\emptyset\}) \geq 1$$

Modal structural rules

$$\mathsf{d}^{[]}\ \frac{\Gamma\{[\emptyset]\}}{\Gamma\{\emptyset\}} \qquad \mathsf{t}^{[]}\ \frac{\Gamma\{[\Delta]\}}{\Gamma\{\Delta\}} \qquad \mathsf{b}^{[]}\ \frac{\Gamma\{[\Sigma, [\Delta]]\}}{\Gamma\{[\Sigma], \Delta\}}$$

$$4^{[]}\ \frac{\Gamma\{[\Delta], [\Sigma]\}}{\Gamma\{[[\Delta], \Sigma]\}} \qquad 5^{[]}\ \frac{\Gamma\{[\Delta]\}\{\emptyset\}}{\Gamma\{\emptyset\}\{[\Delta]\}}\ \mathit{depth}(\Gamma\{\ \}\{[\Delta]\}) \geq 1$$

Classical Rules: Example

$$5 : \Diamond A \supset \Box \Diamond A$$

Classical Rules: Example

$$\dfrac{\Box\bar{A}, \Box\Diamond A}{5 : \Diamond A \supset \Box\Diamond A}\vee$$

Classical Rules: Example

$$\cfrac{\cfrac{\Box\bar{A}, [\Diamond A]}{\Box\bar{A}, \Box\Diamond A}\ \Box}{5 : \Diamond A \supset \Box\Diamond A}\ \vee$$

Classical Rules: Example

$$\vee\ \dfrac{\Box\ \dfrac{\text{cut}\ \dfrac{\Box\bar{A}, [\Diamond\Diamond A, \Diamond A] \qquad \Box\bar{A}, [\Box\Box\bar{A}, \Diamond A]}{\Box\bar{A}, [\Diamond A]}}{\Box\bar{A}, \Box\Diamond A}}{5 : \Diamond A \supset \Box\Diamond A}$$

Classical Rules: Example

$$
\vee\,\dfrac{\Box\,\dfrac{\mathsf{cut}\,\dfrac{\mathsf{b}^{\Diamond}\,\dfrac{\Box\bar{A}, \Diamond A, [\Diamond A]}{\Box\bar{A}, [\Diamond\Diamond A, \Diamond A]} \qquad \Box\bar{A}, [\Box\Box\bar{A}, \Diamond A]}{\Box\bar{A}, [\Diamond A]}}{\Box\bar{A}, \Box\Diamond A}}{5 : \Diamond A \supset \Box\Diamond A}
$$

Classical Rules: Example

$$\vee\ \dfrac{\Box\ \dfrac{\mathsf{cut}\ \dfrac{\mathsf{b}^{\Diamond}\ \dfrac{\Box\bar{A}, \Diamond A, [\Diamond A]}{\Box\bar{A}, [\Diamond\Diamond A, \Diamond A]} \qquad \Box\ \dfrac{\Box\bar{A}, [[\Box\bar{A}], \Diamond A]}{\Box\bar{A}, [\Box\Box\bar{A}, \Diamond A]}}{\Box\bar{A}, [\Diamond A]}}{\Box\bar{A}, \Box\Diamond A}}{5 : \Diamond A \supset \Box\Diamond A}$$

Classical Rules: Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\Box\bar{A}, \Diamond A, [\Diamond A]}{\Box\bar{A}, [\Diamond\Diamond A, \Diamond A]}\,\mathsf{b}^{\Diamond} \qquad \dfrac{\dfrac{\Box\bar{A}, [[\Box\bar{A}, \Diamond A]]}{\Box\bar{A}, [[\Box\bar{A}], \Diamond A]}\,4^{\Diamond}}{\Box\bar{A}, [\Box\Box\bar{A}, \Diamond A]}\,\Box}{\Box\bar{A}, [\Diamond A]}\,\mathsf{cut}}{\Box\bar{A}, \Box\Diamond A}\,\Box}{5 : \Diamond A \supset \Box\Diamond A}\,\vee$$

Intuitionistic Rules

System NIK

Intuitionistic Rules

System NIK

$$\text{id}\ \frac{}{\Gamma\{a\ , a\ \}}$$

$$\wedge\ \frac{\Gamma\{A\ \} \quad \Gamma\{B\ \}}{\Gamma\{A \wedge B\ \}}$$

$$\vee\ \frac{\Gamma\{A\ \}}{\Gamma\{A \vee B\ \}} \qquad \vee\ \frac{\Gamma\{B\ \}}{\Gamma\{A \vee B\ \}}$$

$$\Box\ \frac{\Gamma\{[A\ ]\}}{\Gamma\{\Box A\ \}} \qquad \Diamond\ \frac{\Gamma\{[A\ , \Delta]\}}{\Gamma\{\Diamond A\ , [\Delta]\}}$$

Intuitionistic Rules

System NIK

$$\mathsf{id}\ \overline{\Gamma\{a\ , a\ \}}$$

$$\wedge\ \frac{\Gamma\{A\ , B\ \}}{\Gamma\{A \wedge B\ \}} \qquad \wedge\ \frac{\Gamma\{A\ \}\quad \Gamma\{B\ \}}{\Gamma\{A \wedge B\ \}}$$

$$\vee\ \frac{\Gamma\{A\ \}\quad \Gamma\{A\ \}}{\Gamma\{A \vee B\ \}} \qquad \vee\ \frac{\Gamma\{A\ \}}{\Gamma\{A \vee B\ \}} \quad \vee\ \frac{\Gamma\{B\ \}}{\Gamma\{A \vee B\ \}}$$

$$\Box\ \frac{\Gamma\{[A\ , \Delta]\}}{\Gamma\{\Box A\ , [\Delta]\}} \quad \Diamond\ \frac{\Gamma\{[A\ ]\}}{\Gamma\{\Diamond A\ \}} \qquad \Box\ \frac{\Gamma\{[A\ ]\}}{\Gamma\{\Box A\ \}} \quad \Diamond\ \frac{\Gamma\{[A\ , \Delta]\}}{\Gamma\{\Diamond A\ , [\Delta]\}}$$

Intuitionistic Rules

System NIK

$$\bot\ \frac{}{\Gamma\{\bot\}} \qquad \mathsf{id}\ \frac{}{\Gamma\{a, a\}}$$

$$\wedge\ \frac{\Gamma\{A, B\}}{\Gamma\{A \wedge B\}} \qquad \wedge\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}}$$

$$\vee\ \frac{\Gamma\{A\} \quad \Gamma\{A\}}{\Gamma\{A \vee B\}} \qquad \vee\ \frac{\Gamma\{A\}}{\Gamma\{A \vee B\}} \quad \vee\ \frac{\Gamma\{B\}}{\Gamma\{A \vee B\}}$$

$$\Box\ \frac{\Gamma\{[A, \Delta]\}}{\Gamma\{\Box A, [\Delta]\}} \quad \Diamond\ \frac{\Gamma\{[A]\}}{\Gamma\{\Diamond A\}} \qquad \Box\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \quad \Diamond\ \frac{\Gamma\{[A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}$$

Intuitionistic Rules

System NIK

$$\bot\ \frac{}{\Gamma\{\bot\}} \qquad \mathsf{id}\ \frac{}{\Gamma\{a, a\}}$$

$$\wedge\ \frac{\Gamma\{A, B\}}{\Gamma\{A \wedge B\}} \qquad \wedge\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}}$$

$$\vee\ \frac{\Gamma\{A\} \quad \Gamma\{A\}}{\Gamma\{A \vee B\}} \qquad \vee\ \frac{\Gamma\{A\}}{\Gamma\{A \vee B\}} \quad \vee\ \frac{\Gamma\{B\}}{\Gamma\{A \vee B\}}$$

$$\supset\ \frac{\Gamma^{\#}\{A\} \quad \Gamma\{B\}}{\Gamma\{A \supset B\}} \qquad \supset\ \frac{\Gamma\{A, B\}}{\Gamma\{A \supset B\}}$$

$$\Box\ \frac{\Gamma\{[A, \Delta]\}}{\Gamma\{\Box A, [\Delta]\}} \quad \Diamond\ \frac{\Gamma\{[A]\}}{\Gamma\{\Diamond A\}} \qquad \Box\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \quad \Diamond\ \frac{\Gamma\{[A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}$$

Intuitionistic Rules

System NIK

$$\bot\ \frac{}{\Gamma\{\bot\}} \qquad \mathsf{c}\ \frac{\Gamma\{A, A\}}{\Gamma\{A\}} \qquad \mathsf{id}\ \frac{}{\Gamma\{a, a\}}$$

$$\wedge\ \frac{\Gamma\{A, B\}}{\Gamma\{A \wedge B\}} \qquad \wedge\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}}$$

$$\vee\ \frac{\Gamma\{A\} \quad \Gamma\{A\}}{\Gamma\{A \vee B\}} \qquad \vee\ \frac{\Gamma\{A\}}{\Gamma\{A \vee B\}} \quad \vee\ \frac{\Gamma\{B\}}{\Gamma\{A \vee B\}}$$

$$\supset\ \frac{\Gamma^{\#}\{A\} \quad \Gamma\{B\}}{\Gamma\{A \supset B\}} \qquad \supset\ \frac{\Gamma\{A, B\}}{\Gamma\{A \supset B\}}$$

$$\Box\ \frac{\Gamma\{[A, \Delta]\}}{\Gamma\{\Box A, [\Delta]\}} \quad \Diamond\ \frac{\Gamma\{[A]\}}{\Gamma\{\Diamond A\}} \qquad \Box\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \quad \Diamond\ \frac{\Gamma\{[A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}$$

Intuitionistic Rules

System NIK

$$\bot\ \frac{}{\Gamma\{\bot\}} \qquad \mathsf{c}\ \frac{\Gamma\{A, A\}}{\Gamma\{A\}} \qquad \mathsf{id}\ \frac{}{\Gamma\{a, a\}}$$

$$\wedge\ \frac{\Gamma\{A, B\}}{\Gamma\{A \wedge B\}} \qquad \wedge\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}}$$

$$\vee\ \frac{\Gamma\{A\} \quad \Gamma\{A\}}{\Gamma\{A \vee B\}} \qquad \vee\ \frac{\Gamma\{A\}}{\Gamma\{A \vee B\}} \qquad \vee\ \frac{\Gamma\{B\}}{\Gamma\{A \vee B\}}$$

$$\supset\ \frac{\Gamma^{\#}\{A\} \quad \Gamma\{B\}}{\Gamma\{A \supset B\}} \qquad \supset\ \frac{\Gamma\{A, B\}}{\Gamma\{A \supset B\}}$$

$$\Box\ \frac{\Gamma\{[A, \Delta]\}}{\Gamma\{\Box A, [\Delta]\}} \qquad \Diamond\ \frac{\Gamma\{[A]\}}{\Gamma\{\Diamond A\}} \qquad \Box\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \Diamond\ \frac{\Gamma\{[A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}$$

Additional structural rules

$$\mathsf{w}\ \frac{\Gamma\{\emptyset\}}{\Gamma\{\Lambda\}} \qquad \mathsf{cut}\ \frac{\Gamma\{A\} \quad \Gamma^{\#}\{A\}}{\Gamma\{\emptyset\}}$$

Intuitionistic Rules

Modal $\Diamond$ -rules	Modal $\Box$ -rules	Modal structural rules
$\mathsf{d}\ \dfrac{\Gamma\{[A]\}}{\Gamma\{\Diamond A\}}$	$\mathsf{d}\ \dfrac{\Gamma\{[A]\}}{\Gamma\{\Box A\}}$	$\mathsf{d}^{[]}\ \dfrac{\Gamma\{[\emptyset]\}}{\Gamma\{\emptyset\}}$
$\mathsf{t}\ \dfrac{\Gamma\{A\}}{\Gamma\{\Diamond A\}}$	$\mathsf{t}\ \dfrac{\Gamma\{A\}}{\Gamma\{\Box A\}}$	$\mathsf{t}^{[]}\ \dfrac{\Gamma\{[\Delta]\}}{\Gamma\{\Delta\}}$
$\mathsf{b}\ \dfrac{\Gamma\{[\Delta], A\}}{\Gamma\{[\Delta, \Diamond A]\}}$	$\mathsf{b}\ \dfrac{\Gamma\{[\Delta], A\}}{\Gamma\{[\Delta, \Box A]\}}$	$\mathsf{b}^{[]}\ \dfrac{\Gamma\{[\Sigma, [\Delta]]\}}{\Gamma\{[\Sigma], \Delta\}}$
$4\ \dfrac{\Gamma\{[\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}$	$4\ \dfrac{\Gamma\{[\Box A, \Delta]\}}{\Gamma\{\Box A, [\Delta]\}}$	$4^{[]}\ \dfrac{\Gamma\{[\Delta], [\Sigma]\}}{\Gamma\{[[\Delta], \Sigma]\}}$
$5\ \dfrac{\Gamma\{\emptyset\}\{\Diamond A\}}{\Gamma\{\Diamond A\}\{\emptyset\}}$	$5\ \dfrac{\Gamma\{\emptyset\}\{\Box A\}}{\Gamma\{\Box A\}\{\emptyset\}}$	$5^{[]}\ \dfrac{\Gamma\{[\Delta]\}\{\emptyset\}}{\Gamma\{\emptyset\}\{[\Delta]\}}$

45-Closure

Not all combination X of the $\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5$ rules can lead to a complete cut-free system $\mathsf{NK} \cup \mathsf{X}^{\Diamond}$ or $\mathsf{NIK} \cup \mathsf{X} \ \ \cup \mathsf{X}\ \ $.

ex: $\{\mathsf{b}, 5\} \vdash 4 : \Diamond\Diamond A \supset \Diamond A$

but 4 is not derivable in $\mathsf{NK} \cup \{\mathsf{b}^{\Diamond}, 5^{\Diamond}\} \setminus \{\mathsf{cut}\}$

If $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$, the 45-closure is defined as:

$$\hat{\mathsf{X}} = \begin{cases} \mathsf{X} \cup \{4\} & \text{if } \{\mathsf{b}, 5\} \subseteq \mathsf{X} \text{ or if } \{\mathsf{t}, 5\} \subseteq \mathsf{X} \\ \mathsf{X} \cup \{5\} & \text{if } \{\mathsf{b}, 4\} \subseteq \mathsf{X} \\ \mathsf{X} & \text{otherwise} \end{cases}$$

Cut Elimination

Theorem: Cut-Elimination in the 45-closure

Let $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$.

- (Brünnler, 2009) If Γ is derivable in $\mathsf{NK} \cup \mathsf{X}^{\Diamond} \cup \{\mathsf{cut}\}$ then it is derivable in $\mathsf{NK} \cup \hat{\mathsf{X}}^{\Diamond}$.
- (Straßburger, 2013) If Γ is derivable in $\mathsf{NIK} \cup \mathsf{X} \;\; \cup \mathsf{X} \;\; \cup \{\mathsf{cut}\}$ then it is derivable in

$$\begin{cases} \mathsf{NIK} \cup \hat{\mathsf{X}} \;\; \cup \hat{\mathsf{X}} & \text{if } \mathsf{d} \notin \mathsf{X} \\ \mathsf{NIK} \cup \hat{\mathsf{X}} \;\; \cup \hat{\mathsf{X}} \;\; \cup \{\mathsf{d}^{[]}\} & \text{if } \mathsf{d} \in \mathsf{X} \end{cases}$$

Modularity

Theorem: Modular Cut-Elimination

Let $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$.

- If Γ is derivable in $\mathsf{NK} \cup \mathsf{X}^{\Diamond} \cup \{\mathsf{cut}\}$ then it is derivable in $\mathsf{NK} \cup \mathsf{X}^{\Diamond} \cup \mathsf{X}^{[]}$.
- If Γ is derivable in $\mathsf{NIK} \cup \mathsf{X} \;\; \cup \mathsf{X} \;\; \cup \{\mathsf{cut}\}$ then it is derivable in $\mathsf{NIK} \cup \mathsf{X} \;\; \cup \mathsf{X} \;\; \cup \mathsf{X}^{[]}$.

Modularity

If Γ is derivable in $\mathsf{NK} \cup \mathsf{X} \cup \{\mathsf{cut}\}$, then we have a proof of Γ in $\mathsf{NK} \cup \hat{\mathsf{X}}^{\Diamond}$.
If $\hat{\mathsf{X}} = \mathsf{X}$, then a proof in $\mathsf{NK} \cup \hat{\mathsf{X}}^{\Diamond}$ is trivially a proof in $\mathsf{NK} \cup \mathsf{X}^{\Diamond} \cup \mathsf{X}^{[]}$.
Otherwise, we must have one of the following three cases:

- If $\{\mathsf{t}, 5\} \subseteq \mathsf{X}$ then $\hat{\mathsf{X}} = \mathsf{X} \cup \{4\}$ ...
- If $\{\mathsf{b}, 5\} \subseteq \mathsf{X}$ then $\hat{\mathsf{X}} = \mathsf{X} \cup \{4\}$ ...
- If $\{\mathsf{b}, 4\} \subseteq \mathsf{X}$ then $\hat{\mathsf{X}} = \mathsf{X} \cup \{5\}$ ...

Modularity

- If $\{\mathsf{t}, \mathsf{5}\} \subseteq \mathsf{X}$ then $\hat{\mathsf{X}} = \mathsf{X} \cup \{\mathsf{4}\}$ and the $\mathsf{4}^{\Diamond}$-rule is admissible in $\mathsf{NK} \cup \mathsf{X}^{\Diamond}$.

$$\mathsf{4}^{\Diamond}\,\dfrac{\Gamma\{[\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} \quad\rightsquigarrow\quad \mathsf{t}^{[]}\,\dfrac{\mathsf{5}^{\Diamond}\,\dfrac{\mathsf{w}\,\dfrac{\Gamma\{[\Diamond A, \Delta]\}}{\Gamma\{[\emptyset], [\Diamond A, \Delta]\}}}{\Gamma\{[\Diamond A], [\Delta]\}}}{\Gamma\{\Diamond A, [\Delta]\}}$$

- If $\{\mathsf{b}, \mathsf{5}\} \subseteq \mathsf{X}$ then $\hat{\mathsf{X}} = \mathsf{X} \cup \{\mathsf{4}\}$ and the $\mathsf{4}^{\Diamond}$-rule is admissible in $\mathsf{NK} \cup \mathsf{X}^{\Diamond}$.

$$\mathsf{4}^{\Diamond}\,\dfrac{\Gamma\{[\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} \quad\rightsquigarrow\quad \mathsf{b}^{[]}\,\dfrac{\mathsf{5}^{\Diamond}\,\dfrac{\mathsf{w}\,\dfrac{\Gamma\{[\Diamond A, \Delta]\}}{\Gamma\{[[\emptyset], \Diamond A, \Delta]\}}}{\Gamma\{[[\Diamond A], \Delta]\}}}{\Gamma\{\Diamond A, [\Delta]\}}$$

Modularity

- If $\{b, 4\} \subseteq X$ then $\hat{X} = X \cup \{5\}$. We replace the $5^\Diamond$ by the equivalent set of rules $\{5^\Diamond_1, 5^\Diamond_2, 5^\Diamond_3\}$ and show that all three are admissible in $NK \cup X^\Diamond$.

$$5^\Diamond_1 \; \frac{\Gamma\{[\Delta], \Diamond A\}}{\Gamma\{[\Delta, \Diamond A]\}} \qquad 5^\Diamond_2 \; \frac{\Gamma\{[\Delta], [\Diamond A, \Sigma]\}}{\Gamma\{[\Delta, \Diamond A], [\Sigma]\}} \qquad 5^\Diamond_3 \; \frac{\Gamma\{[\Delta, [\Diamond A, \Sigma]]\}}{\Gamma\{[\Delta, \Diamond A, [\Sigma]]\}}$$

Concluding Remarks

- We used both logical and structural rules to get a modular cut-free system but for some combinations of axioms only the structural or the logical rules would be sufficient depending on the system.
- In order to better understand this phenomenon, we need to find a general pattern for translating axioms into rules and to investigate for which type of axioms such a translation is possible.