Modular Focused Proof Systems for Intuitionistic Modal Logics

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Abstract

Focusing is a general technique for syntactically compartmentalizing the non-deterministic choices in a proof system, which not only improves proof search but also has the representational benefit of distilling sequent proofs into synthetic normal forms. However, since focusing is usually specified as a restriction of the sequent calculus, the technique has not been transferred to logics that lack a (shallow) sequent presentation, as is the case for some of the logics of the modal cube. We have recently extended the focusing technique to classical nested sequents, a generalization of ordinary sequents. In this work we further extend focusing to intuitionistic nested sequents, which can capture all the logics of the intuitionistic S5 cube in a modular fashion. We present an internal cut-elimination procedure for the focused system which in turn is used to show its completeness.

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1 Introduction

When one adds features to a proof system, one generally expects that the meta-theory of the system becomes more complicated. Take, for example, the one-sided sequent calculus G3c for classical propositional logic, which has just as many logical rules as connectives and two additional structural rules of identity and cut. Eliminating cuts from this system is relatively straightforward: there is a single cut rule and a simple lexicographic induction on the cut rank and heights of derivations. If we move to a system with multiplicative rules and structural rules of contraction and weakening, such as Gentzen's original system LK, then the single cut rule and lexicographic measure is no longer sufficient to handle permutations of cuts with contraction and weakening: we either need to add additional rules such as mix or we need to use a sophisticated induction measure that takes the number of contractions on the cut formula into account. Extending the system with the modal connectives $\Box$ and $\Diamond$ and the modal axiom k adds new forms of cuts and further complications to the measure. Adding other modal axioms such as t or 4 causes new structural rules to appear that now need to be considered for cut permutations. Other modal axioms such as 5 causes the very structure of (list-like) sequents to no longer be adequate for building analytic proof systems, so the notion of sequent needs to be generalized, say to *labeled sequents* [15, 20] or to *hypersequents* [2]. Needless to say, the cut rules for such generalized sequents—indeed, the entire cut-elimination procedure—needs to be re-examined. Finally, moving to a two-sided sequent calculus, which is essential for intuitionistic versions of these calculi, doubles the number of inference rules, and hence doubles the number of cases to consider in the cut-elimination argument.

In this paper, we report on some observations that seem to suggest that this trend of increasing syntactic and meta-theoretic complexity can be halted and even reversed if

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one designs the proof system to enforce certain normal forms. We use the intuitionistic propositional logics of the modal S5 cube (see Figure 1) as our testbed, as it contains all permutations of the complications mentioned in the previous paragraph. We did not begin this work with the goal of improving the proof systems for such logics; we were instead interested in the pragmatic question of automated proof search in modal logics, in both their classical and intuitionistic dialects. For such logics, proof systems based on *nested sequents*, a generalization of the usual list-like sequents (as formulated by Gentzen) to tree-like structures [4], turn out to have certain desirable properties from the perspective of proof search. Specifically, (1) they are *analytic*, meaning that every theorem can be proved using only sequents built from subformulas of the theorem; (2) their meta-theory is *internal*, which means that procedures such as cut-elimination operate directly on the proofs in the system rather than by translation to a different system; and (3) they are *modular*, which means that the axes of extension in the modal cube correspond exactly to the choice of specific inference rules. This final desideratum of modularity turns out to be fairly non-trivial [4, 5, 12].

One direct way to improve proof search is to reduce the *proof search space*, which lets a search procedure make fewer choices to get farther. Over the past two decades, the *focusing* technique, originally developed for (linear) logic programming [14, 1] has turned out to be a generally applicable method of reducing the proof search space that remains complete (*i.e.*, every theorem has a focused proof). It has been transplanted from its origin in the sequent calculus for linear logic [1] to a wide variety of logics [8, 11, 17] and proof systems [6, 3, 7], and it is empirically a very "high impact" optimization to standard proof search procedures [8, 13]. This generality suggests that the ability to transform a proof system into a focused form is a good indication of its syntactic quality, in a manner similar to how admissibility of cut shows that a proof system is syntactically consistent.

We have recently shown how to adapt the focusing technique to the classical nested proof systems [7]. Now, nested proof systems also exist for the intuitionistic versions of these modal systems [21, 12], so it is natural to ask if our focusing technique applies here as well. The intuitionistic restriction in these systems is achieved by means of an *input/output annotation* on the formulas that corresponds to whether that formula is a hypothesis or a conclusion [9]. As long as there is exactly one output in a sequent, its semantic meaning is intuitionistic, and hence the inference rules of the system are designed to preserve this singular occurrence of output formulas. These annotations cause every rule corresponding to connectives and the modal axioms to have two incarnations, one for an input-annotated and the other for an output-annotated formula. Section 3 summarizes the system NIK from [21] that we use as the basis of our focused systems.

Our starting point, therefore, was a focused version of the proof system containing annotated formulas. However, we were surprised to discover that: (1) the input/output annotations turn out to be redundant, as they can always be uniquely inferred; and that (2) in the *synthetic* form (Section 5) of the system, which elides the details of the focused logical rules and records only the *phase transitions*, there is only a single modal *structural* rule that is needed for every axiom. It turns out that the synthetic version of the system has *fewer* structural rules than the non-focused version, and the same number of structural rules as the classical system, which we did not expect would be the case. The input/output annotations are shown to be unnecessary by the use of *polarized* syntax that separates the classes of *positive* formulas, whose output rules are non-invertible (therefore requiring non-deterministic choices during search) and *negative* formulas, whose input rules are non-invertible. We can show that intuitionistically meaningful polarized sequents are exactly those sequents with a single negative formula interpreted as the output.

Like in the classical case [7], our main technical contribution is the proof of completeness of the focused calculus by means of an internal cut-elimination proof. In the process of writing this proof, we discovered a further simplification of the synthetic version of the focused system: the so called *store* rule of focused calculi [7, Figure 3] that we used earlier in the classical system is also unnecessary. Indeed, removing the store rule makes the *decision* and *release* rules of the system correspond exactly to the introduction rules for the two shift connectives $\downarrow$ and $\uparrow$, respectively, that inject each polarized class into the other. This simplification in turn makes the three cuts that were required in the classical cut-elimination argument [7, Figures 6 and 9] merely variants of a single cut rule. Moreover, since this simplification was effectively independent of the classical or intuitionistic flavor of the logic, we observe exactly the same reduction of the number of cut rules to just a single rule in the intuitionistic synthetic system described in Section 5. We then obtain a cut-elimination proof (Theorem 6.6)—and its corollary, the completeness of the synthetic system—that is considerably *shorter* and *simpler*, using a more standard induction measure, than the corresponding proof in [7].

Besides these technical contributions, we would like to stress the following conceptual point: focusing, written in a synthetic form, is not a complication one *adds* to a proof system and its associated meta-theory, but a *simplification* of both. Such a simplification has already been observed for ordinary intuitionistic logic by Zeilberger [22]. As we add more features to a logic, the effect of this simplification becomes more noticeable.

2 Preliminaries on Intuitionistic Modal Logic

We will work with the following grammar of *formulas* (written $A, B, \ldots$), which are built from a collection of *atomic formulas* (written $a, b, \ldots$).

$$A, B, \ldots ::= a \mid A \wedge B \mid \top \mid A \vee B \mid \bot \mid A \supset B \mid \Box A \mid \Diamond A$$

This grammar is slightly redundant because $\top$ can be defined as $a \supset a$ for some atom a. We nevertheless keep it in the syntax because one of the polarized versions of $\top$, which we will encounter in Section 4, will turn out to be non-redundant. Recall that classical modal logic K is obtained from classical propositional logic by adding to any standard formulation, such as Hilbert's axiomatization,

- a *necessitation* rule that says that $\Box A$ is a theorem of K if A is a theorem; and
- the axiom of *distributivity*, commonly called k: $\Box(A \supset B) \supset (\Box A \supset \Box B)$.

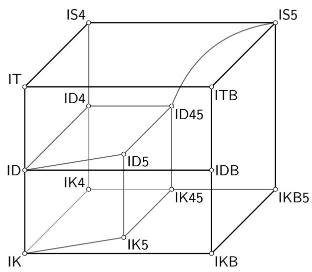
Obtaining the intuitionistic variant of K is more involved. Lacking De Morgan duality, there are several variants of k that are classically but not intuitionistically equivalent. In this paper, we consider the intuitionistic variant of the modal logic K, called IK, that is obtained from ordinary intuitionistic propositional logic (IPL) by

- adding the *necessitation rule*: $\Box A$ is a theorem of IK if A is a theorem; and
- adding the following five variants of the k axiom.

$$\begin{array}{lll} \mathsf{k}_1 : \Box(A \supset B) \supset (\Box A \supset \Box B) & \mathsf{k}_3 : \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B) & \mathsf{k}_5 : \Diamond\bot \supset \bot \\ \mathsf{k}_2 : \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) & \mathsf{k}_4 : (\Diamond A \supset \Box B) \supset \Box(A \supset B) & \end{array} \tag{1}$$

This logic IK was first studied in [18] and [16], and then was investigated in detail in [20], particularly its standard Kripke semantics based on birelational models.

In this paper, we will also examine the intuitionistic variants of the axioms d, t, b, $\mathsf{4}$, and $\mathsf{5}$ that are shown on the right in Figure 1. As in the classical case, they give rise to 15



$\mathsf{d}\colon \Box A \supset \Diamond A$	(Seriality)
$\mathsf{t}\colon (A \supset \Diamond A) \wedge (\Box A \supset A)$	(Reflexivity)
$\mathsf{b}\colon (A \supset \Box\Diamond A) \wedge (\Diamond\Box A \supset A)$	(Symmetry)
$\mathsf{4}\colon (\Diamond\Diamond A \supset \Diamond A) \wedge (\Box A \supset \Box\Box A)$	(Transitivity)
$\mathsf{5}\colon (\Diamond A \supset \Box\Diamond A) \wedge (\Diamond\Box A \supset \Box A)$	(Euclideanness)

Figure 1 The intuitionistic modal S5 cube and the five constituent axioms.

different distinct logics that can be arranged in a cube, the so-called *S5-cube*. (There are fewer than 32 logics because of redundant sets such as $\{\mathsf{t}, \mathsf{5}\}$ and $\{\mathsf{b}, \mathsf{4}\}$ that both yield the logic $\mathsf{IS5}$.) The intuitionistic variant of the cube is shown on the left in Figure 1.

For a given set $\mathsf{X} \subseteq \{\mathsf{t}, \mathsf{d}, \mathsf{4}, \mathsf{b}, \mathsf{5}\}$, we write $\mathsf{IK}{+}\mathsf{X}$ for the logic that is obtained from IK by adding the axioms in X. A formula A is said to be X-*valid* iff it is a theorem of $\mathsf{IK}{+}\mathsf{X}$.[1] In addition, we define the *45-closure* of X, denoted by $\hat{\mathsf{X}}$, as follows:

$$\hat{\mathsf{X}} = \begin{cases} \mathsf{X}{+}\mathsf{4} & \text{if } \{\mathsf{b}, \mathsf{5}\} \subseteq \mathsf{X} \text{ or if } \{\mathsf{t}, \mathsf{5}\} \subseteq \mathsf{X} \\ \mathsf{X}{+}\mathsf{5} & \text{if } \{\mathsf{b}, \mathsf{4}\} \subseteq \mathsf{X} \\ \mathsf{X} & \text{otherwise} \end{cases}$$

If $\mathsf{X} = \hat{\mathsf{X}}$ we also say that X is *45-closed*. In this case we have that whenever the $\mathsf{4}$ axiom (or the $\mathsf{5}$ axiom) is derivable in $\mathsf{IK}{+}\mathsf{X}$, then $\mathsf{4}$ (or $\mathsf{5}$ resp.) is already contained in X. Every logic in the cube in Figure 1 can be defined by at least one 45-closed set of axioms [4].

3 Intuitionistic Modal Logic in Nested Sequents

This section is a summary of the nested sequent system NIK from [21]. The standard formulation of NIK is based closely on the classical system KN [7, 4]. A *nested sequent* is a finite tree where each node contains a multiset of formulas. In the classical case, this tree is then endowed with an *interpretation* where, at each node, the interpretation of each child subtree is *boxed* (using $\Box$) and considered to be disjunctively related to that of the other child subtrees and to the formulas at the node. This interpretation is purely symmetric. To move to the intuitionistic case, we need to introduce an essential asymmetry between the *input* (*i.e.*, the *left*) formulas, which constitute the hypotheses, and the singleton *output* (or the *right*) that constitutes the conclusion. Exactly one of the formulas in the tree will therefore be annotated with a special mark, depicted with a superscript $^\circ$, to signify that it is the output; all other formulas will then be interpreted as inputs.

To be concrete, we will present nested sequents in terms of a grammar of *input sequents* (written Λ) where the output formula does not occur, and *full sequents* (written Γ) where the output formula does occur. When the distinction between input and full sequents is not essential, we will use Δ to stand for either case. The relationship between parent and child in the tree will be represented using *bracketing* ([ ]).

$$\Lambda ::= \emptyset \mid A, \Lambda \mid [\Lambda_1], \Lambda_2 \qquad\qquad \Gamma ::= \Lambda, A^\circ \mid \Lambda, [\Gamma] \qquad\qquad \Delta ::= \Lambda \mid \Gamma$$

[1] We slightly abuse the term *valid* as we do not refer to semantics in this paper.

Every full sequent Γ therefore has the shape $\Lambda_1, [\Lambda_2, \cdots [\Lambda_n, A^\circ] \cdots]$. As usual, we consider sequents to be identical up to a reordering of the comma-separated elements. Observe that removing the output formula from a full sequent yields an input sequent. We write Λ, Δ to stand for the concatenation of Λ and Δ, given inductively by $\emptyset, \Delta = \Delta$; $(A, \Lambda), \Delta = A, (\Lambda, \Delta)$; and $([\Lambda_1], \Lambda_2), \Delta = [\Lambda_1], (\Lambda_2, \Delta)$.

▶ **Definition 3.1** (Meaning). The *meaning* of a NIK sequent Δ is a formula, written $\mathrm{fm}(\Delta)$, that obeys the following equations.

$$\mathrm{fm}(\emptyset) = \top \qquad \mathrm{fm}(A, \Lambda) = A \wedge \mathrm{fm}(\Lambda) \qquad \mathrm{fm}([\Lambda_1], \Lambda_2) = \Diamond \mathrm{fm}(\Lambda_1) \wedge \mathrm{fm}(\Lambda_2)$$
$$\mathrm{fm}(\Lambda, A^\circ) = \mathrm{fm}(\Lambda) \supset A \qquad \mathrm{fm}(\Lambda, [\Gamma]) = \mathrm{fm}(\Lambda) \supset \Box \mathrm{fm}(\Gamma)$$

We assume that any occurrences of $A \wedge \top$ and $\top \supset A$ in the meaning are simplified to A. ◀

▶ **Example 3.2.** Consider the following full sequent: $\Gamma = A, B, [C^\circ, [B]], [D, A, [C]]$. It is considered identical to $A, B, [D, A, [C]], [[B], C^\circ]$ and represents this tree:

$$\begin{array}{ccc} & A, B & \\ / & & \backslash \\ C^\circ & & D, A \\ | & & | \\ B & & C \end{array}$$

We also have $\mathrm{fm}(\Gamma) = A \wedge B \wedge \Diamond(D \wedge A \wedge \Diamond C) \supset \Box(\Diamond B \supset C)$. ◀

The inference rules for nested sequents will operate on subtrees of such sequents. To identify such subtrees, we use the notions of *contexts* and *substitutions*.

▶ **Definition 3.3** (Context). An *n-holed context* is like a sequent but contains n pairwise distinct numbered *holes* of the form $\{\,\}_i$ (for $i \in 1..n$) wherever an input formula may otherwise occur. We depict such a context as $\Delta\{\,\}_1 \cdots \{\,\}_n$. Given n sequents $\Delta_1, \ldots, \Delta_n$ (called the *arguments*), we write $\Delta\{\Delta_1\} \cdots \{\Delta_n\}$, called a *substitution*, to stand for the sequent where the hole $\{\,\}_i$ in $\Delta\{\,\}_1 \cdots \{\,\}_n$ has been replaced by Δ_i (for $i \in 1..n$), assuming that the result is well-formed, *i.e.*, there is at most one $^\circ$-annotated formula. Note that if $\Delta_i = \emptyset$ we simply remove the hole $\{\,\}_i$. A *full context* is a context of the form $\Gamma\{\,\}_1 \cdots \{\,\}_n$, which means that there is an output formula in $\Gamma\{\emptyset\}_1 \cdots \{\emptyset\}_n$. Thus, all the arguments to this context must be input sequents. On the other hand, an *input context* is of the form $\Lambda\{\,\}_1 \cdots \{\,\}_n$ and contains only input formulas, so when it is used to build a sequent at most one of its arguments can itself be a full sequent. ◀

In the rest of this paper, we will omit the hole index subscripts (except when there is some ambiguity) to keep the notation light. Note that a 0-holed context is the same as a sequent. Given a 1-holed context that contains no output formulas, *i.e.*, of the form $\Lambda\{\,\}$, it is permissible to replace the hole with a full sequent Γ, in which case the substitution $\Lambda\{\Gamma\}$ is also a full sequent. If the context contains an output formula, however, then this formula must be removed before such a substitution is syntactically well-formed.

▶ **Definition 3.4** (Output Deletion). We write $\Delta^*\{\,\}_1 \cdots \{\,\}_n$ for the result of deleting any output formulas from an n-holed context $\Delta\{\,\}_1 \cdots \{\,\}_n$. ◀

▶ **Example 3.5.** Consider $\Lambda\{\,\} = [[B, C], \{\,\}], C$; $\Gamma_1\{\,\} = C, [\{\,\}, [B, C^\circ]]$; and $\Gamma_2 = A, [B^\circ]$. Then, $\Lambda\{\Gamma_2\} = C, [[B, C], A, [B^\circ]]$ and $\Gamma_1\{\Lambda\{\emptyset\}\} = C, [[[B, C]], C, [B, C^\circ]]$. $\Gamma_1\{\Gamma_2\}$ is not well-formed because it would contain both C° and B°, but $\Gamma_1^*\{\Gamma_2\} = C, [[B], A, [B^\circ]]$. ◀

NIK

$$
\mathsf{id}\,\frac{}{\Lambda\{a,a^\circ\}}
\qquad
\wedge_{\mathsf{R}}\,\frac{\Lambda\{A^\circ\}\quad\Lambda\{B^\circ\}}{\Lambda\{A\wedge B^\circ\}}
\qquad
\wedge_{\mathsf{L}}\,\frac{\Gamma\{A,B\}}{\Gamma\{A\wedge B\}}
\qquad
\top_{\mathsf{R}}\,\frac{}{\Lambda\{\top^\circ\}}
$$

$$
\vee_{\mathsf{R1}}\,\frac{\Lambda\{A^\circ\}}{\Lambda\{A\vee B^\circ\}}
\qquad
\vee_{\mathsf{R2}}\,\frac{\Lambda\{B^\circ\}}{\Lambda\{A\vee B^\circ\}}
\qquad
\vee_{\mathsf{L}}\,\frac{\Gamma\{A\}\quad\Gamma\{B\}}{\Gamma\{A\vee B\}}
\qquad
\bot_{\mathsf{L}}\,\frac{}{\Gamma\{\bot\}}
$$

$$
\supset_{\mathsf{R}}\,\frac{\Lambda\{A,B^\circ\}}{\Lambda\{A\supset B^\circ\}}
\qquad
\supset_{\mathsf{L}}\,\frac{\Gamma^*\{A\supset B,A^\circ\}\quad\Gamma\{B\}}{\Gamma\{A\supset B\}}
$$

$$
\Diamond_{\mathsf{Rk}}\,\frac{\Lambda_1\{[\Lambda_2,A^\circ]\}}{\Lambda_1\{[\Lambda_2],\Diamond A^\circ\}}
\qquad
\Diamond_{\mathsf{L}}\,\frac{\Gamma\{[A]\}}{\Gamma\{\Diamond A\}}
\qquad
\Box_{\mathsf{R}}\,\frac{\Lambda\{[A^\circ]\}}{\Lambda\{\Box A^\circ\}}
\qquad
\Box_{\mathsf{Lk}}\,\frac{\Delta_1\{\Box A,[A,\Delta_2]\}}{\Delta_1\{\Box A,[\Delta_2]\}}
$$

X

$$
\Diamond_{\mathsf{Rt}}\,\frac{\Lambda\{A^\circ\}}{\Lambda\{\Diamond A^\circ\}}
\qquad
\Box_{\mathsf{Lt}}\,\frac{\Gamma\{\Box A,A\}}{\Gamma\{\Box A\}}
\qquad
\Diamond_{\mathsf{R4}}\,\frac{\Lambda_1\{[\Lambda_2,\Diamond A^\circ]\}}{\Lambda_1\{[\Lambda_2],\Diamond A^\circ\}}
\qquad
\Box_{\mathsf{L4}}\,\frac{\Delta_1\{\Box A,[\Box A,\Delta_2]\}}{\Delta_1\{\Box A,[\Delta_2]\}}
$$

$$
\Diamond_{\mathsf{Rd}}\,\frac{\Lambda\{[A^\circ]\}}{\Lambda\{\Diamond A^\circ\}}
\qquad
\Box_{\mathsf{Ld}}\,\frac{\Gamma\{\Box A,[A]\}}{\Gamma\{\Box A\}}
\qquad
\Diamond_{\mathsf{Rb}}\,\frac{\Lambda_1\{[\Lambda_2],A^\circ\}}{\Lambda_1\{[\Lambda_2,\Diamond A^\circ]\}}
\qquad
\Box_{\mathsf{Lb}}\,\frac{\Delta_1\{[\Box A,\Delta_2],A\}}{\Delta_1\{[\Box A,\Delta_2]\}}
$$

$$
\Diamond_{\mathsf{R5}}\,\frac{\Lambda\{\emptyset\}\{\Diamond A^\circ\}}{\Lambda\{\Diamond A^\circ\}\{\emptyset\}}\ \mathrm{dp}(\Lambda\{\ \}\{\emptyset\})>0
\qquad
\Box_{\mathsf{L5}}\,\frac{\Gamma\{\Box A\}\{\Box A\}}{\Gamma\{\Box A\}\{\emptyset\}}\ \mathrm{dp}(\Gamma\{\ \}\{\emptyset\})>0
$$

Figure 2 The NIK+X family of nested sequent systems for intuitionistic modal logics

We now have enough ingredients to define the inference rules for NIK, which are displayed in Figure 2. The rules in the upper box are common to every logic in the modal cube and so we call just this core system NIK. For every collection of axioms $\mathsf{X} \subseteq \{\mathsf{t},\mathsf{d},\mathsf{4},\mathsf{b},\mathsf{5}\}$, we define the system NIK+X by adding to NIK the rules $\Diamond_{\mathsf{Rx}}$ and $\Box_{\mathsf{Lx}}$ for every $\mathsf{x} \in \mathsf{X}$. Note that in the rules $\Box_{\mathsf{Lk}}$, $\Box_{\mathsf{L4}}$, $\Box_{\mathsf{Lb}}$ exactly one of the $\Delta_1\{\ \}$ and Δ_2 is a full sequent (context), and the other is an input sequent (context), as only one of them can contain the unique output formula.

The $\Diamond_{\mathsf{R5}}$ and $\Box_{\mathsf{L5}}$ rules have a side condition on the *depth* of the occurrence of the principal formula, which must not be in the root of the tree representation of the sequent. This is a direct consequence of the fact that the 5 axiom implies that $\Diamond\cdots\Diamond A\supset\Box\Diamond A$, so a bracketed $\Diamond A^\circ$ in the conclusion can be derived from any $\Diamond A^\circ$ under a prefix of n $\Diamond$s, which can then be moved into any other bracket at depth n in the premise using $\Diamond_{\mathsf{Rk}}$.

▶ **Definition 3.6** (Depth). The *depth* of a 1-holed context $\Delta\{\ \}$, written $\mathrm{dp}(\Delta\{\ \})$, is given inductively by $\mathrm{dp}(\{\ \}) = 0$; $\mathrm{dp}(\Delta_1,\Delta_2\{\ \}) = \mathrm{dp}(\Delta_2\{\ \})$; $\mathrm{dp}(\Delta_1,[\Delta_2\{\ \}]) = 1+\mathrm{dp}(\Delta_2\{\ \})$. ◀

▶ **Example 3.7.** We give as an example the proof of $\mathsf{k_4} : (\Diamond p\supset\Box n)\supset\Box(p\supset n)$ in NIK.

$$
\supset_{\mathsf{R}}\dfrac{\Box_{\mathsf{R}}\dfrac{\supset_{\mathsf{R}}\dfrac{\supset_{\mathsf{L}}\dfrac{\Diamond_{\mathsf{R}}\dfrac{\mathsf{id}\dfrac{}{\Diamond p\supset\Box n,[p^\circ,p]}}{\Diamond p\supset\Box n,\Diamond p^\circ,[p]}\qquad \Box_{\mathsf{L}}\dfrac{\mathsf{id}\dfrac{}{\Box n,[p,n,n^\circ]}}{\Box n,[p,n^\circ]}}{\Diamond p\supset\Box n,[p,n^\circ]}\,(\dagger)}{\Diamond p\supset\Box n,[p\supset n^\circ]}}{\Diamond p\supset\Box n,\Box(p\supset n)^\circ}}{(\Diamond p\supset\Box n)\supset\Box(p\supset n)^\circ}
$$

Observe how n° is deleted from the first premise of (†). ◀

The following theorem summarizes the main results of [21].

▶ **Theorem 3.8.** *Let* $\mathsf{X} \subseteq \{\mathsf{t},\mathsf{d},\mathsf{4},\mathsf{b},\mathsf{5}\}$ *be 45-closed and let* Γ *be a sequent. The following are equivalent:*

1. $\mathrm{fm}(\Gamma)$ *is* X*-valid.*
2. Γ *is provable in* NIK+X+cut*, where* cut *is:* $\mathsf{cut}\,\dfrac{\Gamma^*\{A^\circ\}\quad\Gamma\{A\}}{\Gamma\{\emptyset\}}$
3. Γ *is provable in* NIK+X*.*

Proof. See [21, Theorems 4.1, 5.1, 5.3, and 6.7]. 45-closure is only needed for cut-elimination and cut-free completeness. It is not necessary for soundness and completeness with cut. ◀

4 Focused Nested Sequents

In the previous section, input and output formulas were differentiated using annotations, but without any particular restrictions on which kinds of formulas may receive which annotations. It turns out that certain connectives are endowed with inherent affinities for one or the other annotation. For instance, $\Diamond$-formulas in the output tend to remain as side formulas until the sequent has the adequate bracketing structure, but input $\Diamond$-formulas can be decomposed eagerly since $\Diamond_{\mathsf{L}}$ is an invertible rule. For example, the sequent $\Diamond a, \Diamond a^{\circ}$ can only be proved by applying the $\Diamond_{\mathsf{L}}$ rule below $\Diamond_{\mathsf{Rk}}$. In the terminology of polarities and focusing, $\Diamond$-formulas are *synchronous* or *positive*.

It turns out that we can classify every formula—not just $\Diamond$—into either a *positive formula*, whose right rules are non-invertible, or a *negative formula*, whose left rules are non-invertible. For nearly every kind of formula, this classification is canonically determined; the exceptions are the atoms, where the choice of polarity is free as long as each atom is assigned exactly one polarity, and the $\wedge$ and $\top$ connectives, which are ambiguous in the sense that it is possible to design inference rules for them that give them a positive or a negative interpretation. Following [7, 11], we divide $\wedge$ and $\top$ into their polarized incarnations as separate connectives; $\wedge$ into its positive and negative polarizations, $\overset{+}{\wedge}$ and $\overset{-}{\wedge}$, and $\top$ into $\overset{+}{\top}$ and $\overset{-}{\top}$. Formulas are therefore divided into the positive (written with P, Q) and negative (written with N, M) classes as follows.

$$P, Q ::= L \mid P \overset{+}{\wedge} Q \mid \overset{+}{\top} \mid P \vee Q \mid \bot \mid \Diamond P \qquad L ::= p \mid {\downarrow} N$$
$$N, M ::= R \mid M \overset{-}{\wedge} N \mid \overset{-}{\top} \mid P \supset N \mid \Box N \qquad R ::= n \mid {\uparrow} P$$

We write L for particular positive formulas that we call *left-neutral formulas*, and R for particular negative formulas that we call *right-neutral formulas*. They can be atoms or built from the polarity *shifts* $\downarrow$ and $\uparrow$, which are used to move between the two polarized classes.

Polarized sequents are similar to NIK sequents, but instead of using annotations, we force input formulas to be positive and output formulas to be negative. The resulting grammar for *polarized input sequents* (written Ω) and *polarized full sequents* (written Σ) is then:

$$\Omega ::= \emptyset \mid P, \Omega \mid [\Omega_1], \Omega_2 \qquad \Sigma ::= \Omega, N \mid \Omega, [\Sigma] \qquad \Theta ::= \Omega \mid \Sigma$$

Observe that in any polarized full sequent there is always exactly one negative formula. In building the focused proof system, we will largely confine ourselves to *neutral input sequents* (written Λ) and *neutral full sequents* (written Γ), which are those subclasses of polarized input sequents and polarized full sequents that are built up of neutral formulas. In other words, they have the following grammar.

$$\Lambda ::= \emptyset \mid L, \Lambda \mid [\Lambda_1], \Lambda_2 \qquad \Gamma ::= \Lambda, R \mid \Lambda, [\Gamma] \qquad \Delta ::= \Lambda \mid \Gamma$$

Let us now give the meanings of these polarized sequents.

▶ **Definition 4.1** (Depolarization). Every polarized formula P or N is related to an unpolarized formula by a *depolarization* map $\lfloor\ \rfloor$ with the following inductive definition.

$$\lfloor p \rfloor = p \quad \lfloor {\downarrow} N \rfloor = \lfloor N \rfloor \qquad \lfloor n \rfloor = n \quad \lfloor {\uparrow} P \rfloor = \lfloor P \rfloor$$
$$\lfloor P \overset{+}{\wedge} Q \rfloor = \lfloor P \rfloor \wedge \lfloor Q \rfloor \quad \lfloor \overset{+}{\top} \rfloor = \top \quad \lfloor P \vee Q \rfloor = \lfloor P \rfloor \vee \lfloor Q \rfloor \quad \lfloor \bot \rfloor = \bot \quad \lfloor \Diamond P \rfloor = \Diamond \lfloor P \rfloor$$

FoNIK

$$\mathsf{id}^{\mathsf{f}}_{\mathsf{R}}\ \frac{}{\Gamma\{p, \langle p\rangle\}} \qquad \overset{+}{\wedge}{}^{\mathsf{f}}_{\mathsf{R}}\ \frac{\Gamma\{\langle P\rangle\} \quad \Gamma\{\langle Q\rangle\}}{\Gamma\{\langle P \mathbin{\overset{+}{\wedge}} Q\rangle\}} \qquad \overset{+}{\top}{}^{\mathsf{f}}_{\mathsf{R}}\ \frac{}{\Gamma\{\langle \overset{+}{\top}\rangle\}} \qquad \vee^{\mathsf{f}}_{\mathsf{R1}}\ \frac{\Gamma\{\langle P\rangle\}}{\Gamma\{\langle P \vee Q\rangle\}} \qquad \vee^{\mathsf{f}}_{\mathsf{R2}}\ \frac{\Gamma\{\langle Q\rangle\}}{\Gamma\{\langle P \vee Q\rangle\}}$$

$$\mathsf{id}^{\mathsf{f}}_{\mathsf{L}}\ \frac{}{\Lambda\{n, \langle n\rangle\}} \qquad \overset{-}{\wedge}{}^{\mathsf{f}}_{\mathsf{L1}}\ \frac{\Gamma\{\langle M\rangle\}}{\Gamma\{\langle M \mathbin{\overset{-}{\wedge}} N\rangle\}} \qquad \overset{-}{\wedge}{}^{\mathsf{f}}_{\mathsf{L2}}\ \frac{\Gamma\{\langle N\rangle\}}{\Gamma\{\langle M \mathbin{\overset{-}{\wedge}} N\rangle\}} \qquad \supset^{\mathsf{f}}_{\mathsf{L}}\ \frac{\Gamma\{\langle P\rangle\} \quad \Gamma\{\langle N\rangle\}}{\Gamma\{\langle P \supset N\rangle\}}$$

$$\Diamond^{\mathsf{f}}_{\mathsf{Rk}}\ \frac{\Delta_1\{[\Delta_2, \langle P\rangle]\}}{\Delta_1\{[\Delta_2], \langle \Diamond P\rangle\}} \qquad \Box^{\mathsf{f}}_{\mathsf{Lk}}\ \frac{\Delta_1\{[\Delta_2, \langle N\rangle]\}}{\Delta_1\{[\Delta_2], \langle \Box N\rangle\}}$$

$$\uparrow^{\mathsf{f}}_{\mathsf{R}}\ \frac{\Lambda\{\uparrow P, \langle P\rangle\}}{\Lambda\{\uparrow P\}} \qquad \uparrow^{\mathsf{f}}_{\mathsf{L}}\ \frac{\Gamma\{P\}}{\Gamma\{\langle \uparrow P\rangle\}} \qquad \downarrow^{\mathsf{f}}_{\mathsf{L}}\ \frac{\Gamma\{\downarrow N, \langle N\rangle\}}{\Gamma\{\downarrow N\}} \qquad \downarrow^{\mathsf{f}}_{\mathsf{R}}\ \frac{\Gamma^*\{N\}}{\Gamma\{\langle \downarrow N\rangle\}}$$

$$\overset{+}{\wedge}{}^{\mathsf{f}}_{\mathsf{L}}\ \frac{\Sigma\{P, Q\}}{\Sigma\{P \mathbin{\overset{+}{\wedge}} Q\}} \qquad \overset{+}{\top}{}^{\mathsf{f}}_{\mathsf{L}}\ \frac{\Sigma\{\emptyset\}}{\Sigma\{\overset{+}{\top}\}} \qquad \vee^{\mathsf{f}}_{\mathsf{L}}\ \frac{\Sigma\{P\} \quad \Sigma\{Q\}}{\Sigma\{P \vee Q\}} \qquad \bot^{\mathsf{f}}_{\mathsf{L}}\ \frac{}{\Sigma\{\bot\}} \qquad \Diamond^{\mathsf{f}}_{\mathsf{L}}\ \frac{\Sigma\{[P]\}}{\Sigma\{\Diamond P\}}$$

$$\overset{-}{\wedge}{}^{\mathsf{f}}_{\mathsf{R}}\ \frac{\Omega\{M\} \quad \Omega\{N\}}{\Omega\{M \mathbin{\overset{-}{\wedge}} N\}} \qquad \overset{-}{\top}{}^{\mathsf{f}}_{\mathsf{R}}\ \frac{}{\Omega\{\overset{-}{\top}\}} \qquad \supset^{\mathsf{f}}_{\mathsf{R}}\ \frac{\Omega\{P, N\}}{\Omega\{P \supset N\}} \qquad \Box^{\mathsf{f}}_{\mathsf{R}}\ \frac{\Omega\{[N]\}}{\Omega\{\Box N\}}$$

X^{f}

$$\Diamond^{\mathsf{f}}_{\mathsf{Rt}}\ \frac{\Gamma\{\langle P\rangle\}}{\Gamma\{\langle \Diamond P\rangle\}} \qquad \Box^{\mathsf{f}}_{\mathsf{Lt}}\ \frac{\Gamma\{\langle N\rangle\}}{\Gamma\{\langle \Box N\rangle\}} \qquad \Diamond^{\mathsf{f}}_{\mathsf{R4}}\ \frac{\Delta_1\{[\Delta_2, \langle \Diamond P\rangle]\}}{\Delta_1\{[\Delta_2], \langle \Diamond P\rangle\}} \qquad \Box^{\mathsf{f}}_{\mathsf{L4}}\ \frac{\Delta_1\{[\Delta_2, \langle \Box N\rangle]\}}{\Delta_1\{[\Delta_2], \langle \Box N\rangle\}}$$

$$\Diamond^{\mathsf{f}}_{\mathsf{Rd}}\ \frac{\Gamma\{[\langle P\rangle]\}}{\Gamma\{\langle \Diamond P\rangle\}} \qquad \Box^{\mathsf{f}}_{\mathsf{Ld}}\ \frac{\Gamma\{[\langle N\rangle]\}}{\Gamma\{\langle \Box N\rangle\}} \qquad \Diamond^{\mathsf{f}}_{\mathsf{Rb}}\ \frac{\Delta_1\{[\Delta_2], \langle P\rangle\}}{\Delta_1\{[\Delta_2, \langle \Diamond P\rangle]\}} \qquad \Box^{\mathsf{f}}_{\mathsf{Lb}}\ \frac{\Delta_1\{[\Delta_2], \langle N\rangle\}}{\Delta_1\{[\Delta_2, \langle \Box N\rangle]\}}$$

$$\Diamond^{\mathsf{f}}_{\mathsf{R5}}\ \frac{\Gamma\{\emptyset\}\{\langle \Diamond P\rangle\}}{\Gamma\{\langle \Diamond P\rangle\}\{\emptyset\}}\ \mathrm{dp}(\Gamma\{\ \}\{\emptyset\}) > 1 \qquad \Box^{\mathsf{f}}_{\mathsf{L5}}\ \frac{\Gamma\{\emptyset\}\{\langle \Box N\rangle\}}{\Gamma\{\langle \Box N\rangle\}\{\emptyset\}}\ \mathrm{dp}(\Gamma\{\ \}\{\emptyset\}) > 1$$

Figure 3 The FoNIK+X$^{\mathsf{f}}$ family, focused versions of NIK+X from Figure 2.

$$\lfloor M \mathbin{\overset{-}{\wedge}} N\rfloor = \lfloor M\rfloor \wedge \lfloor N\rfloor \qquad \lfloor \overset{-}{\top}\rfloor = \top \qquad \lfloor P \supset N\rfloor = \lfloor P\rfloor \supset \lfloor N\rfloor \qquad \lfloor \Box N\rfloor = \Box\lfloor N\rfloor$$

We say that P or N is X-*valid* iff $\lfloor P\rfloor$ or $\lfloor N\rfloor$ is X-valid, respectively. ◀

▶ **Definition 4.2** (Meaning). The *meaning* of a polarized sequent Θ, written $\mathrm{fm}(\Theta)$, is a positive or a negative formula (respectively) obeying:

$$\mathrm{fm}(\emptyset) = \overset{+}{\top} \qquad \mathrm{fm}(P, \Omega) = P \mathbin{\overset{+}{\wedge}} \mathrm{fm}(\Omega) \qquad \mathrm{fm}([\Omega_1], \Omega_2) = \Diamond \mathrm{fm}(\Omega_1) \mathbin{\overset{+}{\wedge}} \mathrm{fm}(\Omega_2)$$

$$\mathrm{fm}(\Omega, N) = \mathrm{fm}(\Omega) \supset N \qquad \mathrm{fm}(\Omega, [\Sigma]) = \mathrm{fm}(\Omega) \supset \Box \mathrm{fm}(\Sigma)$$ ◀

▶ **Definition 4.3** (Polarized Context). An *n-holed polarized context* is like a polarized sequent but contains n pairwise distinct numbered *holes* of the form $\{\ \}_i$ (for $i \in 1..n$) wherever a positive formula may otherwise occur. We depict such a context as $\Theta\{\ \}_1 \cdots \{\ \}_n$. Given n polarized sequents $\Theta_1, \ldots, \Theta_n$ (the *arguments*), we write the *substitution* $\Theta\{\Theta_1\} \cdots \{\Theta_n\}$ to mean the sequent where the hole $\{\ \}_i$ in $\Theta\{\ \}_1 \cdots \{\ \}_n$ is replaced by Θ_i (or removed if $\Theta_i = \emptyset$), for $i \in 1..n$, assuming that the result is well-formed, *i.e.*, that there is at most one negative formula in the result. We write $\Theta^*\{\ \}_1 \cdots \{\ \}_n$ for the context formed by deleting any negative formula from $\Theta\{\ \}_1 \cdots \{\ \}_n$.[2] ◀

The inference system for polarized sequents will be *focused* [1]. A focused proof is a proof where the *decision* to apply a non-invertible rule to a neutral formula must be explicitly taken, and this decision commits the proof to retaining focus on its transitive subformulas until there is a polarity change. This focusing protocol drastically reduces the space of proofs, since rules can only be applied to the focused formula when one exists. Nevertheless, every derivable polarized sequent has a focused proof, as we will see in Section 6.

[2] We reuse the right-deletion notation from Definition 3.4 in the polarized case since the concepts are similar, replacing "output formula" with "negative formula".

▶ **Definition 4.4** (Focused Sequent)**.** A *focused sequent* is like a neutral sequent but contains an additional single occurrence of $\langle P\rangle$ or $\langle N\rangle$ wherever a positive formula may otherwise occur, called its *focus*. We depict such sequents as $\Gamma\{\langle P\rangle\}$ or $\Gamma\{\langle N\rangle\}$ where $\Gamma\{\,\}$ is a neutral context (*i.e.*, $\Gamma\{\emptyset\}$ is a neutral sequent). The meaning of a focused sequent is written by extending $\mathrm{fm}(\;)$, which now obeys $\mathrm{fm}(\Gamma\{\langle P\rangle\}) = \mathrm{fm}(\Gamma^*\{\uparrow P\})$ and $\mathrm{fm}(\Gamma\{\langle N\rangle\}) = \mathrm{fm}(\Gamma\{\downarrow N\})$. ◀

The inference rules of the family of focused sequent systems $\mathsf{FoNIK{+}X^f}$ are given in Figure 3. Like with $\mathsf{NIK{+}X}$ earlier, for any $\mathsf{X} \subseteq \{\mathsf{t}, \mathsf{d}, \mathsf{4}, \mathsf{b}, \mathsf{5}\}$, we define $\mathsf{FoNIK{+}X^f}$ to be the system FoNIK, consisting of the rules in the upper section of Figure 3, extended with $\Box^{\mathsf{f}}_{\mathsf{Lx}}$ and $\Diamond^{\mathsf{f}}_{\mathsf{Rx}}$ (for each $\mathsf{x} \in \mathsf{X}$) in the lower section of the figure.

A focused proof of a neutral sequent begins—reading from conclusion upwards—with a neutral end-sequent, to which only the two rules $\downarrow^{\mathsf{f}}_{\mathsf{L}}$ and $\uparrow^{\mathsf{f}}_{\mathsf{R}}$ may be applied. In each case a neutral shifted formula is selected for focus, at which point the proof enters the *focused* phase, which persists until the focus again becomes neutral. At this point, the proof either finishes with $\mathsf{id}^{\mathsf{f}}_{\mathsf{R}}$ or $\mathsf{id}^{\mathsf{f}}_{\mathsf{L}}$ if the focus is atomic, or it enters the *active* phase using the rules $\uparrow^{\mathsf{f}}_{\mathsf{L}}$ or $\downarrow^{\mathsf{f}}_{\mathsf{R}}$. Note that, because the $\downarrow^{\mathsf{f}}_{\mathsf{R}}$ rule introduces a negative formula to the premise sequent, any other existing negative formulas must be deleted. In the active phase, positive and negative formulas are decomposed, in an arbitrary order, using left and right rules respectively, until eventually the sequent becomes neutral again.

▶ **Example 4.5.** Let $R = \uparrow\downarrow(\downarrow(\Diamond p \supset \Box n) \supset \Box(p \supset n))$ which is a right-neutral polarized form of $\mathsf{k_4}$ (see (1)) with a minimal number of shifts. Below is the derivation of R in FoNIK, and therefore the focused version of the derivation in Example 3.7:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$\mathsf{id}^{\mathsf{f}}_{\mathsf{R}}$}
\UnaryInfC{$R, \downarrow(\Diamond p \supset \Box n), [\langle p\rangle, p, n]$}
\RightLabel{$\Diamond^{\mathsf{f}}_{\mathsf{Rk}}$}
\UnaryInfC{$R, \downarrow(\Diamond p \supset \Box n), \langle \Diamond p\rangle, [p, n]$}
\AxiomC{}
\RightLabel{$\mathsf{id}^{\mathsf{f}}_{\mathsf{L}}$}
\UnaryInfC{$R, \downarrow(\Diamond p \supset \Box n), [\langle n\rangle, p, n]$}
\RightLabel{$\Box^{\mathsf{f}}_{\mathsf{Lk}}$}
\UnaryInfC{$R, \downarrow(\Diamond p \supset \Box n), \langle \Box n\rangle, [p, n]$}
\RightLabel{$\supset^{\mathsf{f}}_{\mathsf{L}}$}
\BinaryInfC{$R, \downarrow(\Diamond p \supset \Box n), \langle \Diamond p \supset \Box n\rangle, [p, n]$}
\RightLabel{$\downarrow^{\mathsf{f}}_{\mathsf{L}}\ (\dagger)$}
\UnaryInfC{$R, \downarrow(\Diamond p \supset \Box n), [p, n]$}
\RightLabel{$\supset^{\mathsf{f}}_{\mathsf{R}}$}
\UnaryInfC{$R, \downarrow(\Diamond p \supset \Box n), [p \supset n]$}
\RightLabel{$\Box^{\mathsf{f}}_{\mathsf{R}}$}
\UnaryInfC{$R, \downarrow(\Diamond p \supset \Box n), \Box(p \supset n)$}
\RightLabel{$\supset^{\mathsf{f}}_{\mathsf{R}}$}
\UnaryInfC{$R, \downarrow(\Diamond p \supset \Box n) \supset \Box(p \supset n)$}
\RightLabel{$\downarrow^{\mathsf{f}}_{\mathsf{R}}$}
\UnaryInfC{$R, \langle\downarrow(\downarrow(\Diamond p \supset \Box n) \supset \Box(p \supset n))\rangle$}
\RightLabel{$\uparrow^{\mathsf{f}}_{\mathsf{R}}$}
\UnaryInfC{R}
\end{prooftree}
$$

Observe that the instance of $\downarrow^{\mathsf{f}}_{\mathsf{L}}$ marked $(\dagger)$ cannot be applied any lower in the derivation, since its conclusion would not then be neutral. ◀

▶ **Lemma 4.6** (Soundness)**.** *Let* $\mathsf{X} \subseteq \{\mathsf{t}, \mathsf{d}, \mathsf{4}, \mathsf{b}, \mathsf{5}\}$*. If* Σ *is provable in* $\mathsf{FoNIK{+}X^f}$ *then it is* X*-valid.*

Proof. The proof makes use of the following two inference rules

$$
\mathsf{weak}\ \frac{\Gamma\{\emptyset\}}{\Gamma\{\Lambda\}} \qquad \text{and} \qquad \mathsf{cont}\ \frac{\Gamma\{\Lambda, \Lambda\}}{\Gamma\{\Lambda\}}
$$

defined on unpolarized sequents, and the fact that they are admissible for $\mathsf{NIK{+}X}$ (see [21, Lemma 6.4]). Now, any polarized sequent Σ can be transformed into an unpolarized sequent $\lfloor\Sigma\rfloor$ with the same meaning by replacing every formula P in Σ by $\lfloor P\rfloor$ and the unique formula N in Σ by $\lfloor N\rfloor^\circ$, and similarly for contexts $\Sigma\{\,\}$. Then we can define $\lfloor\Sigma\{\langle N\rangle\}\rfloor = \lfloor\Sigma\rfloor\{\lfloor N\rfloor\}$ and $\lfloor\Sigma\{\langle P\rangle\}\rfloor = \lfloor\Sigma\rfloor^*\{\lfloor P\rfloor^\circ\}$. Every rule in $\mathsf{FoNIK{+}X^f}$ then either becomes trivial or can be simulated by a derivation consisting of an instance of a rule in $\mathsf{NIK{+}X}$ and an instance of weak, except for $\uparrow^{\mathsf{f}}_{\mathsf{R}}$ and $\downarrow^{\mathsf{f}}_{\mathsf{L}}$, which become instances of cont. Thus, a proof of Σ in $\mathsf{FoNIK{+}X^f}$ is transformed into a proof of $\lfloor\Sigma\rfloor$ in $\mathsf{NIK{+}X} + \mathsf{weak} + \mathsf{cont}$. The lemma now follows from admissibility of weak and cont for $\mathsf{NIK{+}X}$ and Theorem 3.8. ◀

5 Synthetic Nested Sequents

Ultimately, we wish to establish the following relation between NIK+X and FoNIK+X$^{\mathsf{f}}$.

▶ **Theorem 5.1** (Soundness and Completeness of FoNIK+X$^{\mathsf{f}}$). *The neutral sequent* Γ *is derivable in* FoNIK+X$^{\mathsf{f}}$ *if and only if* $\lfloor\Gamma\rfloor$ *is derivable in* NIK+X*.*

However, directly showing this statement is rather complicated because of the number of rules in FoNIK+X$^{\mathsf{f}}$. The issue is actually worse than it appears since we would like to have the completeness of focusing be a consequence of cut-elimination and identity reduction in FoNIK+X$^{\mathsf{f}}$, following a strategy initially described by Laurent [10] that has turned out to be remarkably versatile [8, 11, 19, 7]. To retrace this meta-theory directly in FoNIK+X$^{\mathsf{f}}$ would require a carefully managed collection of cuts with a lengthy and intricate argument.

Can the system be simplified? Since the boundary rules $\downarrow^{\mathsf{f}}_{\mathsf{L}}$, $\uparrow^{\mathsf{f}}_{\mathsf{R}}$, $\uparrow^{\mathsf{f}}_{\mathsf{L}}$ and $\downarrow^{\mathsf{f}}_{\mathsf{R}}$ are limited to conclusions that are either neutral or focused, we can see a FoNIK+X$^{\mathsf{f}}$ derivation as progressing in large *synthetic* steps where the rest of the rules are elided. In this section, we give a presentation of FoNIK+X$^{\mathsf{f}}$ that formally builds only such synthetic derivations. Importantly, the synthetic system has far fewer rules than FoNIK+X$^{\mathsf{f}}$. In particular, there is no longer a duplication of the modal rules into $\Diamond^{\mathsf{f}}_{\mathsf{R}}$ and $\Box^{\mathsf{f}}_{\mathsf{L}}$ versions. Nevertheless, this system will be sound and complete with respect to both NIK+X and FoNIK+X$^{\mathsf{f}}$, giving us Theorem 5.1 as a corollary.

The basis of the synthetic system is to isolate the subformula relation and generalize it into an inductively defined *substructure relation*, written $\Subset$, that determines, for a given focus, what formulas would be present in the fringe of the focused phase rooted on it. Since only neutral formulas occur at the fringes, these substructures would be made up of neutral formulas. The inductive definition of the subformula relation is given in the uppermost part of Figure 4. When $\Lambda \Subset P$ or $\Gamma \Subset N$, we say that Λ or Γ is, respectively, a *synthetic substructure* of P or N. Intuitively, each substructure defines a particular collection of *disjunctive* choices available in a corresponding focused phase. The focused phase is launched from a neutral sequent by picking a suitable neutral formula for focus, selecting one of its substructures, and then *contextualizing* the substructure using a generalization of focused sequents (Definition 4.4).

▶ **Definition 5.2** (Contextualizing Sequent). A *contextualizing sequent* is like a neutral sequent but contains a single occurrence of a *focus* of the form $\langle\Delta\rangle$ (where Δ is a neutral sequent) where a positive neutral formula may otherwise occur. Such sequents are written as $\Gamma\{\langle\Delta\rangle\}$ where $\Gamma\{\emptyset\}$ is a neutral sequent. The *meaning* of a contextualizing sequent is written using $\mathrm{fm}(\,)$ obeying: $\mathrm{fm}(\Gamma\{\langle\Lambda\rangle\}) = \mathrm{fm}(\Gamma^*\{\uparrow\mathrm{fm}(\Lambda)\})$ and $\mathrm{fm}(\Gamma_1\{\langle\Gamma_2\rangle\}) = \mathrm{fm}(\Gamma_1\{\downarrow\mathrm{fm}(\Gamma_2)\})$. ◀

The synthetic system SyNIK will be built using neutral and contextualizing sequents. The rules of SyNIK+X$^{\langle\rangle}$ (for any $\mathsf{X} \subseteq \{\mathsf{t}, \mathsf{d}, 4, \mathsf{b}, 5\}$) are shown in Figure 4. As before for NIK+X and FoNIK+X$^{\mathsf{f}}$, we define SyNIK+X$^{\langle\rangle}$ to be SyNIK extended with $\mathsf{x}^{\langle\rangle}$ for every $\mathsf{x} \in \mathsf{X}$. The $\downarrow^{\langle\rangle}_{\mathsf{L}}$ and $\uparrow^{\langle\rangle}_{\mathsf{R}}$ rules are similar to the $\downarrow^{\mathsf{f}}_{\mathsf{L}}$ and $\uparrow^{\mathsf{f}}_{\mathsf{R}}$ rules from FoNIK, except that, instead of granting focus to the P or N (respectively), one of its substructures is selected for contextualization. The contextualization rules consist of the rules $\{\mathsf{spl}^{\langle\rangle}, \mathsf{fin}^{\langle\rangle}, \mathsf{k}^{\langle\rangle}, \mathsf{t}^{\langle\rangle}, 4^{\langle\rangle}, \mathsf{d}^{\langle\rangle}, \mathsf{b}^{\langle\rangle}, 5^{\langle\rangle}\}$ that serve to divide up or move the focus among the premises of the rule. To prevent needless looping, the $\mathsf{spl}^{\langle\rangle}$ rule has a side condition that neither of the foci in the premises is empty; the $\mathsf{fin}^{\langle\rangle}$ rule handles the empty focus case instead. The modal rules require the focus in the conclusion to be bracketed. Observe that there is exactly one modal rule for every modal axiom, unlike FoNIK+X$^{\mathsf{f}}$ that needed both left and right versions. The $5^{\langle\rangle}$ rule has the usual side condition that $\mathrm{dp}(\Gamma\{\ \}\{\emptyset\}) > 0$.

SyNIK

$$\frac{}{L \Subset L} \qquad \frac{\Lambda_1 \Subset P \quad \Lambda_2 \Subset Q}{\Lambda_1, \Lambda_2 \Subset P \stackrel{+}{\wedge} Q} \qquad \frac{}{\emptyset \Subset \stackrel{+}{\top}} \qquad \frac{\Lambda \Subset P}{\Lambda \Subset P \vee Q} \qquad \frac{\Lambda \Subset Q}{\Lambda \Subset P \vee Q} \qquad \frac{\Lambda \Subset P}{[\Lambda] \Subset \Diamond P}$$

$$\frac{}{R \Subset R} \qquad \frac{\Gamma \Subset M}{\Gamma \Subset M \bar{\wedge} N} \qquad \frac{\Gamma \Subset N}{\Gamma \Subset M \bar{\wedge} N} \qquad \frac{\Lambda \Subset P \quad \Gamma \Subset N}{\Lambda, \Gamma \Subset P \supset N} \qquad \frac{\Gamma \Subset N}{[\Gamma] \Subset \Box N}$$

$$\mathsf{id}^{\langle\rangle}\,\frac{}{\Delta\{a, \langle a \rangle\}} \qquad \uparrow_{\mathsf{L}}^{\langle\rangle}\,\frac{\big\{\Gamma\{\Lambda\} : \Lambda \Subset P\big\}}{\Gamma\{\langle \uparrow P \rangle\}} \qquad \uparrow_{\mathsf{R}}^{\langle\rangle}\,\frac{\Lambda_2 \Subset P \quad \Lambda_1\{\uparrow P, \langle \Lambda_2 \rangle\}}{\Lambda_1\{\uparrow P\}}$$

$$\mathsf{fin}^{\langle\rangle}\,\frac{}{\Gamma\{\langle \emptyset \rangle\}} \qquad \downarrow_{\mathsf{R}}^{\langle\rangle}\,\frac{\big\{\Gamma_1^*\{\Gamma_2\} : \Gamma_2 \Subset N\big\}}{\Gamma_1\{\langle \downarrow N \rangle\}} \qquad \downarrow_{\mathsf{L}}^{\langle\rangle}\,\frac{\Gamma_2 \Subset N \quad \Gamma_1\{\downarrow N, \langle \Gamma_2 \rangle\}}{\Gamma_1\{\downarrow N\}}$$

$$\mathsf{k}^{\langle\rangle}\,\frac{\Delta_1\{[\Delta_2, \langle \Delta_3 \rangle]\}}{\Delta_1\{[\Delta_2], \langle [\Delta_3] \rangle\}} \qquad \mathsf{spl}^{\langle\rangle}\,\frac{\Gamma\{\langle \Delta \rangle\} \quad \Gamma\{\langle \Lambda \rangle\}}{\Gamma\{\langle \Delta, \Lambda \rangle\}}\ \#\Delta > 0,\ \#\Lambda > 0$$

$\mathsf{X}^{\langle\rangle}$

$$\mathsf{t}^{\langle\rangle}\,\frac{\Gamma\{\langle \Delta \rangle\}}{\Gamma\{\langle [\Delta] \rangle\}} \quad \mathsf{4}^{\langle\rangle}\,\frac{\Delta_1\{[\Delta_2, \langle [\Delta_3] \rangle]\}}{\Delta_1\{[\Delta_2], \langle [\Delta_3] \rangle\}} \quad \mathsf{d}^{\langle\rangle}\,\frac{\Gamma\{[\langle \Delta \rangle]\}}{\Gamma\{\langle [\Delta] \rangle\}} \quad \mathsf{b}^{\langle\rangle}\,\frac{\Delta_1\{[\Delta_2], \langle \Delta_3 \rangle\}}{\Delta_1\{[\Delta_2, \langle [\Delta_3] \rangle]\}} \quad \mathsf{5}^{\langle\rangle}\,\frac{\Gamma\{\langle [\Delta] \rangle\}\{\emptyset\}}{\Gamma\{\emptyset\}\{\langle [\Delta] \rangle\}}$$

Figure 4 The SyNIK+X$^{\langle\rangle}$ family, synthetic versions of FoNIK+X$^{\mathsf{f}}$ (Figure 3)

Once the focus has been reduced to a single formula by the other rules, it must either be atomic or a shifted formula. In the former case, we apply the id$^{\langle\rangle}$ rule, which is the common form of id$^{\mathsf{f}}_{\mathsf{R}}$ and id$^{\mathsf{f}}_{\mathsf{L}}$ from FoNIK. When the focus is a shifted formula, we use $\uparrow_{\mathsf{L}}^{\langle\rangle}$ or $\downarrow_{\mathsf{R}}^{\langle\rangle}$; in each case, we iterate over the substructures of the principal formula, producing one premise per substructure. When the substructure is a neutral sequent, we need to remove the right-neutral formula from the surrounding context; this only happens in the $\downarrow_{\mathsf{R}}^{\langle\rangle}$ rule.

▶ **Example 5.3.** Here is the synthetic version of the derivation in Example 4.5.

$$\dfrac{\mathsf{k}^{\langle\rangle}\dfrac{\mathsf{id}^{\langle\rangle}\dfrac{}{R, \downarrow(\Diamond p \supset \Box n), [\langle p \rangle, p, n]}}{R, \downarrow(\Diamond p \supset \Box n), \langle [p] \rangle, [p, n]} \qquad \mathsf{k}^{\langle\rangle}\dfrac{\mathsf{id}^{\langle\rangle}\dfrac{}{R, \downarrow(\Diamond p \supset \Box n), [\langle n \rangle, p, n]}}{R, \downarrow(\Diamond p \supset \Box n), \langle [n] \rangle, [p, n]}}{\uparrow_{\mathsf{R}}^{\langle\rangle}\dfrac{\downarrow_{\mathsf{R}}^{\langle\rangle}\dfrac{\downarrow_{\mathsf{L}}^{\langle\rangle}\dfrac{R, \downarrow(\Diamond p \supset \Box n), \langle [p], [n] \rangle, [p, n]}{R, \downarrow(\Diamond p \supset \Box n), [p, n]}\,(\ddagger)}{R, \langle \downarrow(\downarrow(\Diamond p \supset \Box n) \supset \Box(p \supset n)) \rangle}}{R}\,(\dagger)}\ \mathsf{spl}^{\langle\rangle}$$

The instance ($\dagger$) of $\uparrow_{\mathsf{R}}^{\langle\rangle}$ is applicable as $L = \downarrow(\Diamond p \supset \Box n) \supset \Box(p \supset n)$ is left-neutral, so $L \Subset L$. Likewise, the instance ($\ddagger$) of $\downarrow_{\mathsf{L}}^{\langle\rangle}$ is applicable since $[p], [n] \Subset \Diamond p \supset \Box n$. ◀

▶ **Lemma 5.4** (Soundness). *Let* X $\subseteq$ {t, d, 4, b, 5} *and let* Γ *be a neutral sequent. If* Γ *is provable in* SyNIK+X$^{\langle\rangle}$*, then it is also provable in* FoNIK+X$^{\mathsf{f}}$*.*

Proof (Sketch). The essential idea is to interpret the SyNIK contextualizing sequent $\Gamma\{\langle \Theta \rangle\}$ as the FoNIK focused sequent $\Gamma\{\langle \mathrm{fm}(\Theta) \rangle\}$. The rules of the former can be simulated by the latter because $\Theta \Subset \mathrm{fm}(\Theta)$. Examples 4.5 and 5.3 illustrates this interpretation. The whole proof then works by induction on the given SyNIK+X$^{\langle\rangle}$ derivation; the $\downarrow_{\mathsf{L}}^{\langle\rangle}$ and $\uparrow_{\mathsf{R}}^{\langle\rangle}$ rules are simulated by repeating the derivation of the substructure in the FoNIK sequent rather than as a side premise. The $\uparrow_{\mathsf{L}}^{\langle\rangle}$ and $\downarrow_{\mathsf{R}}^{\langle\rangle}$ rules are easily simulated since the active rules of FoNIK are precisely matched by the $\Subset$ inferences. The x$^{\langle\rangle}$ rules are simulated by $\Diamond^{\mathsf{f}}_{\mathsf{Rx}}$ or $\Box^{\mathsf{f}}_{\mathsf{Lx}}$ rules, respectively depending on whether the focus contains a negative formula or not. Finally, spl$^{\langle\rangle}$ and fin$^{\langle\rangle}$ are simulated by $\stackrel{+}{\wedge}_{\mathsf{R}}^{\mathsf{f}}$ and $\stackrel{+}{\top}_{\mathsf{R}}^{\mathsf{f}}$ respectively. ◀

The less trivial converse of Lemma 5.4 will follow from cut-elimination in the next section.

$$\mathsf{weak}\ \frac{\Gamma\{\emptyset\}}{\Gamma\{\Lambda\}} \qquad \mathsf{k}^{[]}\ \frac{\Delta_1\{[\Delta_2],[\Delta_3]\}}{\Delta_1\{[\Delta_2,\Delta_3]\}} \qquad \mathsf{d}^{[]}\ \frac{\Delta_1\{[\,]\}}{\Delta_1\{\emptyset\}} \qquad \mathsf{b}^{[]}\ \frac{\Delta_1\{[\Delta_2,[\Delta_3]]\}}{\Delta_1\{[\Delta_2],\Delta_3\}}$$

$$\mathsf{cont}\ \frac{\Gamma\{p,p\}}{\Gamma\{p\}} \qquad \mathsf{4}^{[]}\ \frac{\Delta_1\{[\Delta_2],[\Delta_3]\}}{\Delta_1\{[[\Delta_2],\Delta_3]\}} \qquad \mathsf{t}^{[]}\ \frac{\Delta_1\{[\Delta_2]\}}{\Delta_1\{\Delta_2\}} \qquad \mathsf{5}^{[]}\ \frac{\Delta_1\{\emptyset\}\{[\Delta_2]\}}{\Delta_1\{[\Delta_2]\}\{\emptyset\}}\ \mathrm{dp}(\Delta_1\{\,\}\{\emptyset\})>0$$

Figure 5 Structural rules admissible in SyNIK+X$^{\langle\rangle}$

6 Synthetic Meta-Theory

In this section we will show that SyNIK+X$^{\langle\rangle}$ extended with a cut rule can simulate NIK+X derivations under a certain interpretation of the annotations. We will then show that the cut rule is admissible in SyNIK+X$^{\langle\rangle}$, thereby concluding that NIK+X rules under that interpretation are admissible in SyNIK+X$^{\langle\rangle}$, *i.e.*, SyNIK+X$^{\langle\rangle}$ is complete with respect to NIK+X. To formulate the cut rule with a minimum of redundancy, we will need to slightly enlarge our notion of contexts and define a pair of *pruning* operations for such contexts.

▶ **Definition 6.1** (Enlarged Contexts and Pruning). In this section, we allow neutral sequents to contain at most one occurrence of a focus $\langle\Delta\rangle$. For a sequent Δ, we write $\Delta^{\not\langle\rangle}$ to prune its focus if there is one (*i.e.*, if $\Delta = \Delta_1\{\langle\Delta_2\rangle\}$ for some $\Delta_1\{\,\}$, then $\Delta^{\not\langle\rangle} = \Delta_1\{\emptyset\}$; otherwise $\Delta^{\not\langle\rangle} = \Delta$). This definition extends straightforwardly to contexts $\Delta\{\,\}$. For a context $\Delta_1\{\,\}$, we write $\Delta_1^{\star}\{\Delta_2\}$ to mean $\Delta_1\{\Delta_2\}$ if Δ_2 is an input sequent, and $\Delta_1^{*}\{\Delta_2\}$ if Δ_2 is a full sequent (see Definition 3.4). ◀

The synthetic cut rule for SyNIK+X$^{\langle\rangle}$ can then be written concisely as follows:

$$\mathsf{cut}^{\langle\rangle}\ \frac{\Gamma^{\star}\{\Delta\} \quad \Gamma^{\not\langle\rangle}\{\langle\Delta\rangle\}}{\Gamma\{\emptyset\}}$$

Before we can show that cut$^{\langle\rangle}$ is admissible, we need to show the admissibility of the structural rules shown in Figure 5 that are used in the cut-elimination proof. Note that the weakening rule weak can be applied only to input contexts and the contraction rule cont only to positive atoms. While the contraction rule could be generalized to any input context, only this instance is needed in the proof. Of course, the notion of contexts and sequents in these structural rules are enlarged in the sense of Definition 6.1.

▶ **Definition 6.2.** The *height* of a derivation $\mathcal{D}$, denoted by $\mathrm{ht}(\mathcal{D})$, is the height of $\mathcal{D}$ when seen as a tree, *i.e.*, the length of the longest branch from the root to a leaf. We say that a rule $\mathsf{r}\,\frac{\Gamma_1}{\Gamma_2}$ is *admissible* for a system S if for every proof $\mathcal{D}_1$ of Γ_1 in S there is a proof $\mathcal{D}_2$ of Γ_2 in S. We say that it is *height-preserving admissible* if additionally $\mathrm{ht}(\mathcal{D}_2) \le \mathrm{ht}(\mathcal{D}_1)$. ◀

▶ **Lemma 6.3** (Admissible Rules). *Let* X $\subseteq \{\mathsf{t},\mathsf{d},\mathsf{4},\mathsf{b},\mathsf{5}\}$ *be* 45*-closed. The rules* weak, cont, *and* k$^{[]}$ *are height-preserving admissible in* SyNIK+X$^{\langle\rangle}$*, and for every* x $\in$ X*, the rule* x$^{[]}$ *is admissible in* SyNIK+X$^{\langle\rangle}$*.*

Proof. The first part is by straightforward induction on the height of the derivation. The second part is less straightforward, but also by induction on the height of the derivation, following [4, Lemma 9]. Below we show the translation of one case from the case analysis in [4] into our setting. The others are similar.

$$\mathsf{4}^{[]}\,\dfrac{\mathsf{b}^{\langle\rangle}\,\dfrac{\dfrac{\mathcal{D}}{\Delta_1\{\langle\Delta_2\rangle,[\Delta_3],[\Delta_4]\}}}{\Delta_1\{[\langle[\Delta_2]\rangle,\Delta_3],[\Delta_4]\}}}{\Delta_1\{[[\langle[\Delta_2]\rangle,\Delta_3],\Delta_4]\}} \quad\rightsquigarrow\quad \mathsf{5}^{\langle\rangle}\,\dfrac{\mathsf{b}^{\langle\rangle}\,\dfrac{\mathsf{4}^{[]}\,\dfrac{\dfrac{\mathcal{D}}{\Delta_1\{\langle\Delta_2\rangle,[\Delta_3],[\Delta_4]\}}}{\Delta_1\{\langle\Delta_2\rangle,[[\Delta_3],\Delta_4]\}}}{\Delta_1\{[\langle[\Delta_2]\rangle,[\Delta_3],\Delta_4]\}}}{\Delta_1\{[[\langle[\Delta_2]\rangle,\Delta_3],\Delta_4]\}}$$

We rely on the crucial fact that $\mathsf{5} \in \mathsf{X}$ when $\{\mathsf{b},\mathsf{4}\} \subseteq \mathsf{X}$, as X is 45-closed. So, this case illustrates why 45-closure is needed, and also shows that the height of the proof can increase for the admissibility of the $\mathsf{x}^{[]}$ rules. ◀

▶ **Definition 6.4.** The *depth* of a polarized formula P or N, denoted by $\mathrm{dp}(P)$ or $\mathrm{dp}(N)$, is inductively given as follows:

$$\begin{aligned} \mathrm{dp}(p) = \mathrm{dp}(n) &= 1 & \mathrm{dp}(P \overset{+}{\wedge} Q) &= \max(\mathrm{dp}(P),\mathrm{dp}(Q))+1\\ \mathrm{dp}(\bot) = \mathrm{dp}(\overset{+}{\top}) = \mathrm{dp}(\overset{-}{\top}) &= 1 & \mathrm{dp}(P \vee Q) &= \max(\mathrm{dp}(P),\mathrm{dp}(Q))+1\\ \mathrm{dp}(\uparrow P) = \mathrm{dp}(\Diamond P) &= \mathrm{dp}(P)+1 & \mathrm{dp}(P \supset N) &= \max(\mathrm{dp}(P),\mathrm{dp}(N))+1\\ \mathrm{dp}(\downarrow N) = \mathrm{dp}(\Box N) &= \mathrm{dp}(N)+1 & \mathrm{dp}(M \overset{-}{\wedge} N) &= \max(\mathrm{dp}(M),\mathrm{dp}(N))+1 \end{aligned}$$

The *rank* of a neutral sequent Δ, denoted by $\mathrm{rk}(\Delta)$, is the multiset of the depths of the formulas in Δ. Formally, it can be defined inductively as follows:

$$\mathrm{rk}(L,\Lambda) = \{\!\{\mathrm{dp}(L)\}\!\} \uplus \mathrm{rk}(\Lambda) \quad \mathrm{rk}(\Lambda,R) = \mathrm{rk}(\Lambda) \uplus \{\!\{\mathrm{dp}(R)\}\!\} \quad \mathrm{rk}(\Lambda,[\Delta]) = \mathrm{rk}(\Lambda) \uplus \mathrm{rk}(\Delta)$$ ◀

▶ **Lemma 6.5** (Cut Reduction)**.** *Let* $\mathsf{X} \subseteq \{\mathsf{t},\mathsf{d},\mathsf{4},\mathsf{b},\mathsf{5}\}$ *be 45-closed. Given a proof that ends:*

$$\mathsf{cut}^{\langle\rangle}\,\dfrac{\dfrac{\mathcal{D}_1}{\Gamma^{\star}\{\Delta\}} \quad \dfrac{\mathcal{D}_2}{\Gamma^{\emptyset}\{\langle\Delta\rangle\}}}{\Gamma\{\emptyset\}}$$

where $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *are in* $\mathsf{SyNIK{+}X}^{\langle\rangle}$*, there is a proof of* $\Gamma\{\emptyset\}$ *in* $\mathsf{SyNIK{+}X}^{\langle\rangle}$*.*

Proof. By lexicographic induction on the tuple $\langle \mathrm{rk}(\Delta), \mathrm{ht}(\mathcal{D}_2), \mathrm{ht}(\mathcal{D}_1)\rangle$, splitting cases on the last rule instances in $\mathcal{D}_1$ and $\mathcal{D}_2$. Note that the last rule in $\mathcal{D}_2$ always applies to the focus $\langle\Delta\rangle$. We will rewrite the derivation, written using $\rightsquigarrow$, by moving the instance of $\mathsf{cut}^{\langle\rangle}$ to a position of strictly lower measure or eliminating it entirely.

- First, let us consider the cases where $\mathcal{D}_2$ ends with a structural rule:

$$\mathsf{cut}^{\langle\rangle}\,\dfrac{\dfrac{\mathcal{D}_1}{\Gamma^{\star}\{\Delta,\Lambda\}} \quad \mathsf{spl}^{\langle\rangle}\,\dfrac{\dfrac{\mathcal{D}_2'}{\Gamma^{\emptyset}\{\langle\Delta\rangle\}} \quad \dfrac{\mathcal{D}_2''}{\Gamma^{\emptyset}\{\langle\Lambda\rangle\}}}{\Gamma^{\emptyset}\{\langle\Delta,\Lambda\rangle\}}}{\Gamma\{\emptyset\}} \quad\rightsquigarrow\quad \mathsf{cut}^{\langle\rangle}\,\dfrac{\mathsf{cut}^{\langle\rangle}\,\dfrac{\dfrac{\mathcal{D}_1}{\Gamma^{\star}\{\Delta,\Lambda\}} \quad \mathsf{weak}\,\dfrac{\dfrac{\mathcal{D}_2'}{\Gamma^{\emptyset}\{\langle\Delta\rangle\}}}{\Gamma^{\emptyset}\{\langle\Delta\rangle,\Lambda\}}}{\Gamma\{\Lambda\}} \quad \dfrac{\mathcal{D}_2''}{\Gamma^{\emptyset}\{\langle\Lambda\rangle\}}}{\Gamma\{\emptyset\}}$$

$$\mathsf{cut}^{\langle\rangle}\,\dfrac{\dfrac{\mathcal{D}_1}{\Delta_1^{\star}\{[\Delta_2^{\star}],[\Delta_3]\}} \quad \mathsf{k}^{\langle\rangle}\,\dfrac{\dfrac{\mathcal{D}_2'}{\Delta_1^{\emptyset}\{[\Delta_2^{\emptyset},\langle\Delta_3\rangle]\}}}{\Delta_1^{\emptyset}\{[\Delta_2^{\emptyset}],\langle[\Delta_3]\rangle\}}}{\Delta_1\{[\Delta_2]\}} \quad\rightsquigarrow\quad \mathsf{cut}^{\langle\rangle}\,\dfrac{\mathsf{k}^{[]}\,\dfrac{\dfrac{\mathcal{D}_1}{\Delta_1^{\star}\{[\Delta_2^{\star}],[\Delta_3]\}}}{\Delta_1^{\star}\{[\Delta_2^{\star},\Delta_3]\}} \quad \dfrac{\mathcal{D}_2'}{\Delta_1^{\emptyset}\{[\Delta_2^{\emptyset},\langle\Delta_3\rangle]\}}}{\Delta_1\{[\Delta_2]\}}$$

$$\mathsf{cut}^{\langle\rangle}\,\dfrac{\dfrac{\mathcal{D}_1}{\Gamma^{\star}\{[\Delta]\}} \quad \mathsf{d}^{\langle\rangle}\,\dfrac{\dfrac{\mathcal{D}_2'}{\Gamma^{\emptyset}\{[\langle\Delta\rangle]\}}}{\Gamma^{\emptyset}\{\langle[\Delta]\rangle\}}}{\Gamma\{\emptyset\}} \quad\rightsquigarrow\quad \mathsf{d}^{[]}\,\dfrac{\mathsf{cut}^{\langle\rangle}\,\dfrac{\dfrac{\mathcal{D}_1}{\Gamma^{\star}\{[\Delta]\}} \quad \dfrac{\mathcal{D}_2'}{\Gamma^{\emptyset}\{[\langle\Delta\rangle]\}}}{\Gamma\{[\emptyset]\}}}{\Gamma\{\emptyset\}}$$

$$\mathsf{cut}^{\langle\rangle}\,\dfrac{\dfrac{\mathcal{D}_1}{\Gamma^{\star}\{[\Delta]\}} \quad \mathsf{t}^{\langle\rangle}\,\dfrac{\dfrac{\mathcal{D}_2'}{\Gamma^{\emptyset}\{\langle\Delta\rangle\}}}{\Gamma^{\emptyset}\{\langle[\Delta]\rangle\}}}{\Gamma\{\emptyset\}} \quad\rightsquigarrow\quad \mathsf{cut}^{\langle\rangle}\,\dfrac{\mathsf{t}^{[]}\,\dfrac{\dfrac{\mathcal{D}_1}{\Gamma^{\star}\{[\Delta]\}}}{\Gamma^{\star}\{\Delta\}} \quad \dfrac{\mathcal{D}_2'}{\Gamma^{\emptyset}\{\langle\Delta\rangle\}}}{\Gamma\{\emptyset\}}$$

$$
\mathsf{cut}^{\langle\rangle}\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Delta_1^\star\{[\Delta_2^\star,[\Delta_3]]\}\end{array} \qquad \mathsf{b}^{\langle\rangle}\dfrac{\begin{array}{c}\mathcal{D}_2'\\ \Delta_1^{\not\langle\rangle}\{[\Delta_2^{\not\langle\rangle}],\langle\Delta_3\rangle\}\end{array}}{\Delta_1^{\not\langle\rangle}\{[\Delta_2^{\not\langle\rangle},\langle[\Delta_3]\rangle]\}}}{\Delta_1\{[\Delta_2]\}}
\quad\rightsquigarrow\quad
\mathsf{cut}^{\langle\rangle}\dfrac{\mathsf{b}^{[]}\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Delta_1^\star\{[\Delta_2^\star,[\Delta_3]]\}\end{array}}{\Delta_1^\star\{[\Delta_2^\star],\Delta_3\}} \qquad \begin{array}{c}\mathcal{D}_2'\\ \Delta_1^{\not\langle\rangle}\{[\Delta_2^{\not\langle\rangle}],\langle\Delta_3\rangle\}\end{array}}{\Delta_1\{[\Delta_2]\}}
$$

$$
\mathsf{cut}^{\langle\rangle}\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Delta_1^\star\{[\Delta_2^\star],[\Delta_3]\}\end{array} \qquad \mathsf{4}^{\langle\rangle}\dfrac{\begin{array}{c}\mathcal{D}_2'\\ \Delta_1^{\not\langle\rangle}\{[\Delta_2^{\not\langle\rangle},\langle[\Delta_3]\rangle]\}\end{array}}{\Delta_1^{\not\langle\rangle}\{[\Delta_2^{\not\langle\rangle}],\langle[\Delta_3]\rangle\}}}{\Delta_1\{[\Delta_2]\}}
\quad\rightsquigarrow\quad
\mathsf{cut}^{\langle\rangle}\dfrac{\mathsf{4}^{[]}\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Delta_1^\star\{[\Delta_2^\star],[\Delta_3]\}\end{array}}{\Delta_1^\star\{[\Delta_2^\star,[\Delta_3]]\}} \qquad \begin{array}{c}\mathcal{D}_2'\\ \Delta_1^{\not\langle\rangle}\{[\Delta_2^{\not\langle\rangle},\langle[\Delta_3]\rangle]\}\end{array}}{\Delta_1\{[\Delta_2]\}}
$$

$$
\mathsf{cut}^{\langle\rangle}\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma^\star\{[\Delta]\}\{\emptyset\}\end{array} \qquad \mathsf{5}^{\langle\rangle}\dfrac{\begin{array}{c}\mathcal{D}_2'\\ \Gamma^{\not\langle\rangle}\{\emptyset\}\{\langle[\Delta]\rangle\}\end{array}}{\Gamma^{\not\langle\rangle}\{\langle[\Delta]\rangle\}\{\emptyset\}}}{\Gamma\{\emptyset\}\{\emptyset\}}
\quad\rightsquigarrow\quad
\mathsf{cut}^{\langle\rangle}\dfrac{\mathsf{5}^{[]}\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma^\star\{[\Delta]\}\{\emptyset\}\end{array}}{\Gamma^\star\{\emptyset\}\{[\Delta]\}} \qquad \begin{array}{c}\mathcal{D}_2'\\ \Gamma^{\not\langle\rangle}\{\emptyset\}\{\langle[\Delta]\rangle\}\end{array}}{\Gamma\{\emptyset\}\{\emptyset\}}
$$

In each case we can apply the induction hypothesis because $\mathrm{ht}(\mathcal{D}_2') < \mathrm{ht}(\mathcal{D}_2)$ and in the first case also $\mathrm{ht}(\mathcal{D}_2'') < \mathrm{ht}(\mathcal{D}_2)$.[3] Note the use of Lemma 6.3.

- If the last rule in $\mathcal{D}_2$ is an axiom, we have one of the following three cases:

$$
\mathsf{cut}^{\langle\rangle}\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma\{\emptyset\}\end{array} \qquad \mathsf{fin}^{\langle\rangle}\dfrac{}{\Gamma^{\not\langle\rangle}\{\langle\emptyset\rangle\}}}{\Gamma\{\emptyset\}}
\;\rightsquigarrow\;
\begin{array}{c}\mathcal{D}_1\\ \Gamma\{\emptyset\}\end{array}
\qquad\qquad
\mathsf{cut}^{\langle\rangle}\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Lambda\{n\}\end{array} \qquad \mathsf{id}^{\langle\rangle}\dfrac{}{\Lambda^{\not\langle\rangle}\{n,\langle n\rangle\}}}{\Lambda\{n\}}
\;\rightsquigarrow\;
\begin{array}{c}\mathcal{D}_1\\ \Lambda\{n\}\end{array}
$$

$$
\mathsf{cut}^{\langle\rangle}\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma\{p,p\}\end{array} \qquad \mathsf{id}^{\langle\rangle}\dfrac{}{\Gamma^{\not\langle\rangle}\{p,\langle p\rangle\}}}{\Gamma\{p\}}
\;\rightsquigarrow\;
\mathsf{cont}\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma\{p,p\}\end{array}}{\Gamma\{p\}}
$$

For the third case we use the admissibility of atomic contraction.

- Finally, the last rule in $\mathcal{D}_2$ can be $\uparrow_{\mathsf{L}}^{\langle\rangle}$ or $\downarrow_{\mathsf{R}}^{\langle\rangle}$, and if at the same time the last rule in $\mathcal{D}_1$ is the corresponding $\downarrow_{\mathsf{L}}^{\langle\rangle}$ or $\uparrow_{\mathsf{R}}^{\langle\rangle}$ on the cut formula (the cut sequent has to be a singleton in that case), we have one of the two principal cases:

$$
\mathsf{cut}^{\langle\rangle}\dfrac{\downarrow_{\mathsf{L}}^{\langle\rangle}\dfrac{\Gamma_2\in N \qquad \begin{array}{c}\mathcal{D}_1'\\ \Gamma_1\{\downarrow N,\langle\Gamma_2\rangle\}\end{array}}{\Gamma_1\{\downarrow N\}} \qquad \downarrow_{\mathsf{R}}^{\langle\rangle}\dfrac{\left\{\begin{array}{c}\mathcal{D}_{\Gamma_2}\\ \Gamma_1^*\{\Gamma_2\}\end{array}\right\}_{\Gamma_2\in N}}{\Gamma_1\{\langle\downarrow N\rangle\}}}{\Gamma_1\{\emptyset\}}
\;\rightsquigarrow\;
\mathsf{cut}^{\langle\rangle}\dfrac{\mathsf{cut}^{\langle\rangle}\dfrac{\begin{array}{c}\mathcal{D}_1'\\ \Gamma_1\{\downarrow N,\langle\Gamma_2\rangle\}\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma_1\{\langle\downarrow N\rangle\}\end{array}}{\Gamma_1\{\langle\Gamma_2\rangle\}} \qquad \begin{array}{c}\mathcal{D}_{\Gamma_2}\\ \Gamma_1^*\{\Gamma_2\}\end{array}}{\Gamma_1\{\emptyset\}}
$$

$$
\mathsf{cut}^{\langle\rangle}\dfrac{\uparrow_{\mathsf{R}}^{\langle\rangle}\dfrac{\Lambda\in P \qquad \begin{array}{c}\mathcal{D}_1'\\ \Gamma^*\{\uparrow P,\langle\Lambda\rangle\}\end{array}}{\Gamma^*\{\uparrow P\}} \qquad \uparrow_{\mathsf{L}}^{\langle\rangle}\dfrac{\left\{\begin{array}{c}\mathcal{D}_{\Lambda}\\ \Gamma\{\Lambda\}\end{array}\right\}_{\Lambda\in P}}{\Gamma\{\langle\uparrow P\rangle\}}}{\Gamma\{\emptyset\}}
\;\rightsquigarrow\;
\mathsf{cut}^{\langle\rangle}\dfrac{\mathsf{cut}^{\langle\rangle}\dfrac{\begin{array}{c}\mathcal{D}_1'\\ \Gamma^*\{\uparrow P,\langle\Lambda\rangle\}\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma\{\langle\uparrow P\rangle\}\end{array}}{\Gamma\{\langle\Lambda\rangle\}} \qquad \begin{array}{c}\mathcal{D}_{\Lambda}\\ \Gamma\{\Lambda\}\end{array}}{\Gamma\{\emptyset\}}
$$

In both cases we have to apply the induction hypothesis twice: first to the upper cut because $\mathrm{ht}(\mathcal{D}_1') < \mathrm{ht}(\mathcal{D}_1)$, and then to the lower cut because $\mathrm{rk}(\Gamma_2) < \mathrm{rk}(\downarrow N)$ and $\mathrm{rk}(\Lambda) < \mathrm{rk}(\uparrow P)$. After the reduction step the focus is not in the same branch any more, so which branch is considered to be $\mathcal{D}_1$ or $\mathcal{D}_2$ may change, but since the rank has decreased strictly this does not affect the inductive argument.

- Of course, when the last rule in $\mathcal{D}_2$ is $\uparrow_{\mathsf{L}}^{\langle\rangle}$ or $\downarrow_{\mathsf{R}}^{\langle\rangle}$, the last rule in $\mathcal{D}_1$ does not need to be the corresponding $\uparrow_{\mathsf{R}}^{\langle\rangle}$ or $\downarrow_{\mathsf{L}}^{\langle\rangle}$ rule. In that case we have a commutative case: the last rule in $\mathcal{D}_1$ is permuted under the cut:

$$
\mathsf{cut}^{\langle\rangle}\dfrac{\mathsf{r}\dfrac{\begin{array}{c}\mathcal{D}_1'\\ \Gamma_1^\star\{\Delta\}\end{array}}{\Gamma^\star\{\Delta\}} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma^{\not\langle\rangle}\{\langle\Delta\rangle\}\end{array}}{\Gamma\{\emptyset\}}
\quad\rightsquigarrow\quad
\mathsf{r}\dfrac{\mathsf{cut}^{\langle\rangle}\dfrac{\begin{array}{c}\mathcal{D}_1'\\ \Gamma_1^\star\{\Delta\}\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma_1^{\not\langle\rangle}\{\langle\Delta\rangle\}\end{array}}{\Gamma_1\{\emptyset\}}}{\Gamma\{\emptyset\}}
$$

---

[3] We abuse the pruning notation in the cases for $\mathsf{k}^{\langle\rangle}$, $\mathsf{b}^{\langle\rangle}$ and $\mathsf{4}^{\langle\rangle}$ by writing $\Delta_1^\star\{[\Delta_2^\star],\{\,\}\}$ or $\Delta_1^\star\{[\Delta_2^\star,\{\,\}]\}$ to denote the pruned context $\Delta^\star\{\,\}$ when $\Delta\{\,\} = \Delta_1\{[\Delta_2],\{\,\}\}$ or $\Delta\{\,\} = \Delta_1\{[\Delta_2,\{\,\}]\}$ respectively.

The situation above applies if r is $\mathsf{k}^{\langle\rangle}$ or any of the $\mathsf{x}^{\langle\rangle}$ rules, because then there is a focus in $\Gamma\{\;\}$ which is moved by r, and we have $\Gamma_1^{\emptyset}\{\;\} = \Gamma^{\emptyset}\{\;\}$. It also applies if r is one of $\downarrow_{\mathsf{L}}^{\langle\rangle}$ or $\uparrow_{\mathsf{R}}^{\langle\rangle}$ because then $\Gamma\{\;\}$ contains no focus and therefore $\Gamma_1^{\emptyset}\{\;\} = \Gamma^{\emptyset}\{\;\}$. If the last rule in $\mathcal{D}_1$ is $\mathsf{spl}^{\langle\rangle}$ the situation is similar, and if it is one of $\mathsf{id}^{\langle\rangle}$ or $\mathsf{fin}^{\langle\rangle}$, then the cut disappears trivially. Note that the last rule in $\mathcal{D}_1$ is not applying to Δ (which is a singleton) because otherwise it would be a principal case. The only nontrivial commutative cases are when the focus in $\Gamma\{\;\}$ is released by the last rule in $\mathcal{D}_1$ which can be either a $\uparrow_{\mathsf{L}}^{\langle\rangle}$ or a $\downarrow_{\mathsf{R}}^{\langle\rangle}$. In the $\uparrow_{\mathsf{L}}^{\langle\rangle}$-case, we can reduce as follows:

$$\mathsf{cut}^{\langle\rangle}\dfrac{\uparrow_{\mathsf{L}}^{\langle\rangle}\dfrac{\left\{\begin{array}{c}\mathcal{D}_\Lambda\\ \Gamma^\star\{\Lambda\}\{\Delta\}\end{array}\right\}_{\Lambda\in P}}{\Gamma^\star\{\langle\uparrow P\rangle\}\{\Delta\}} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma\{\emptyset\}\{\langle\Delta\rangle\}\end{array}}{\Gamma\{\langle\uparrow P\rangle\}\{\emptyset\}} \quad\rightsquigarrow\quad \uparrow_{\mathsf{L}}^{\langle\rangle}\dfrac{\left\{\mathsf{cut}^{\langle\rangle}\dfrac{\begin{array}{c}\mathcal{D}_\Lambda\\ \Gamma^\star\{\Lambda\}\{\Delta\}\end{array} \qquad \mathsf{weak}\dfrac{\begin{array}{c}\mathcal{D}_2\\ \Gamma\{\emptyset\}\{\langle\Delta\rangle\}\end{array}}{\Gamma\{\Lambda\}\{\langle\Delta\rangle\}}}{\Gamma\{\Lambda\}\{\emptyset\}}\right\}_{\Lambda\in P}}{\Gamma\{\langle\uparrow P\rangle\}\{\emptyset\}}$$

and we only need height-preserving admissibility of weakening in order to apply the induction hypothesis, using $\mathrm{ht}(\mathcal{D}_\Lambda) < \mathrm{ht}(\mathcal{D}_1)$. In the $\downarrow_{\mathsf{R}}^{\langle\rangle}$-case we need to distinguish whether Δ is of the form $\uparrow P$ or $\downarrow N$. In the first case the cut disappears:

$$\mathsf{cut}^{\langle\rangle}\dfrac{\downarrow_{\mathsf{R}}^{\langle\rangle}\dfrac{\left\{\begin{array}{c}\mathcal{D}_{\Gamma_2}\\ \Gamma_1^*\{\Gamma_2\}\{\emptyset\}\end{array}\right\}_{\Gamma_2\in N}}{\Gamma_1^*\{\langle\downarrow N\rangle\}\{\uparrow P\}} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma_1\{\emptyset\}\{\langle\uparrow P\rangle\}\end{array}}{\Gamma_1\{\langle\downarrow N\rangle\}\{\emptyset\}} \quad\rightsquigarrow\quad \downarrow_{\mathsf{R}}^{\langle\rangle}\dfrac{\left\{\begin{array}{c}\mathcal{D}_{\Gamma_2}\\ \Gamma_1^*\{\Gamma_2\}\{\emptyset\}\end{array}\right\}_{\Gamma_2\in N}}{\Gamma_1\{\langle\downarrow N\rangle\}\{\emptyset\}}$$

and in the second we again use height-preserving admissibility of weakening in order to apply the induction hypothesis, as $\mathrm{ht}(\mathcal{D}_{\Gamma_2}) < \mathrm{ht}(\mathcal{D}_1)$:

$$\mathsf{cut}^{\langle\rangle}\dfrac{\downarrow_{\mathsf{R}}^{\langle\rangle}\dfrac{\left\{\begin{array}{c}\mathcal{D}_{\Gamma_2}\\ \Gamma_1^*\{\Gamma_2\}\{\downarrow M\}\end{array}\right\}_{\Gamma_2\in N}}{\Gamma_1\{\langle\downarrow N\rangle\}\{\downarrow M\}} \qquad \downarrow_{\mathsf{R}}^{\langle\rangle}\dfrac{\left\{\begin{array}{c}\mathcal{D}_{\Gamma_3}\\ \Gamma_1^*\{\emptyset\}\{\Gamma_3\}\end{array}\right\}_{\Gamma_3\in M}}{\Gamma_1\{\emptyset\}\{\langle\downarrow M\rangle\}}}{\Gamma_1\{\langle\downarrow N\rangle\}\{\emptyset\}}$$

$$\rightsquigarrow\quad \downarrow_{\mathsf{R}}^{\langle\rangle}\dfrac{\left\{\mathsf{cut}^{\langle\rangle}\dfrac{\begin{array}{c}\mathcal{D}_{\Gamma_2}\\ \Gamma_1^*\{\Gamma_2\}\{\downarrow M\}\end{array} \qquad \downarrow_{\mathsf{R}}^{\langle\rangle}\dfrac{\left\{\mathsf{weak}\dfrac{\begin{array}{c}\mathcal{D}_{\Gamma_3}\\ \Gamma_1^*\{\emptyset\}\{\Gamma_3\}\end{array}}{\Gamma_1^*\{\Gamma_2^*\}\{\Gamma_3\}}\right\}_{\Gamma_3\in M}}{\Gamma_1^*\{\Gamma_2\}\{\langle\downarrow M\rangle\}}}{\Gamma_1^*\{\Gamma_2\}\{\emptyset\}}\right\}_{\Gamma_2\in N}}{\Gamma_1\{\langle\downarrow N\rangle\}\{\emptyset\}}$$

◀

▶ **Theorem 6.6** (Cut-Elimination). *Let* $\mathsf{X} \subseteq \{\mathsf{t},\mathsf{d},\mathsf{4},\mathsf{b},\mathsf{5}\}$ *be 45-closed. If a sequent* Γ *is provable in* $\mathsf{SyNIK{+}X^{\langle\rangle}{+}cut}$*, then it is also provable in* $\mathsf{SyNIK{+}X^{\langle\rangle}}$.

Proof. By induction on the number of cuts in the proof, by repeatedly applying Lemma 6.5, always starting with a topmost cut. ◀

▶ **Lemma 6.7** (Identity). *The following rule is derivable in* SyNIK*:* $\mathsf{sid}^{\langle\rangle}\dfrac{}{\Delta_1\{\Delta_2,\langle\Delta_2\rangle\}}$.

Proof. The proof, by induction on the structure of the focus, is similar to the one for classical modal logics in [7, Lemma 4.6]. ◀

▶ **Lemma 6.8** (Simulation). *Let* A° *be provable in* $\mathsf{NIK{+}X}$*, and let* R *be a right-neutral formula with* $\lfloor R\rfloor = A$*. Then* R *is provable in* $\mathsf{SyNIK{+}X^{\langle\rangle}{+}cut}$.

Proof. First, any NIK sequent can be transformed into a neutral polarized sequent with the same meaning. The connectives are turned into their polarized variant and in particular, a polarity is arbitrarily chosen for every atom, every $\top$, and every $\wedge$; then shifts are added as needed to produce well-formed polarized formulas. Once the formulas are polarized, one can obtain neutrality, and remove the $\circ$-annotation, by adding extra shifts in front of each formula in the sequent as follows: if P is a positive formula, $P \mapsto {\downarrow}{\uparrow}P$ and $P^\circ \mapsto {\uparrow}P$, and if N is a negative formula, $N \mapsto {\downarrow}N$ and $N^\circ \mapsto {\uparrow}{\downarrow}N$. Each rule of NIK+X can therefore be considered as a rule between neutral polarized sequents. As such, it can be shown to be derivable in SyNIK+X$^{\langle\rangle}$+cut. We show the cases for the rules id, $\wedge_\mathsf{L}$ and $\Diamond_{\mathsf{Rd}}$. The other cases are similar.

$$\mathsf{id}\,\dfrac{}{\Lambda\{a,a^\circ\}}\quad\text{becomes}\quad \downarrow_\mathsf{L}^{\langle\rangle}\dfrac{\mathsf{sid}^{\langle\rangle}\dfrac{}{\Lambda\{{\downarrow}{\uparrow}p,\langle{\uparrow}p\rangle,{\uparrow}p\}}}{\Lambda\{{\downarrow}{\uparrow}p,{\uparrow}p\}}\quad\text{or}\quad \uparrow_\mathsf{R}^{\langle\rangle}\dfrac{\mathsf{sid}^{\langle\rangle}\dfrac{}{\Lambda\{{\downarrow}n,{\uparrow}{\downarrow}n,\langle{\downarrow}n\rangle\}}}{\Lambda\{{\downarrow}n,{\uparrow}{\downarrow}n\}}$$

$$\wedge_\mathsf{L}\,\dfrac{\Gamma\{A,B\}}{\Gamma\{A\wedge B\}}\quad\text{becomes}$$

$$\mathsf{cut}^{\langle\rangle}\dfrac{\mathsf{weak}\,\dfrac{\Gamma\{{\downarrow}{\uparrow}P,{\downarrow}{\uparrow}Q\}}{\Gamma\{{\downarrow}{\uparrow}(P\overset{+}{\wedge}Q),{\downarrow}{\uparrow}P,{\downarrow}{\uparrow}Q\}}\qquad \mathsf{spl}^{\langle\rangle}\dfrac{\downarrow_\mathsf{R}^{\langle\rangle}\dfrac{\downarrow_\mathsf{L}^{\langle\rangle}\dfrac{\downarrow_\mathsf{R}^{\langle\rangle}\dfrac{\left\{\uparrow_\mathsf{R}^{\langle\rangle}\dfrac{\mathsf{sid}^{\langle\rangle}\dfrac{}{\Gamma^*\{{\downarrow}{\uparrow}(P\overset{+}{\wedge}Q),\Delta_P,\Delta_Q,{\uparrow}Q,\langle\Delta_Q\rangle\}}}{\Gamma^*\{{\downarrow}{\uparrow}(P\overset{+}{\wedge}Q),\Delta_P,\Delta_Q,{\uparrow}Q\}}\right\}_{\Delta_P\in P,\Delta_Q\in Q}}{\Gamma^*\{{\downarrow}{\uparrow}(P\overset{+}{\wedge}Q),\langle{\uparrow}(P\overset{+}{\wedge}Q)\rangle,{\uparrow}Q\}}}{\Gamma^*\{{\downarrow}{\uparrow}(P\overset{+}{\wedge}Q),{\uparrow}Q\}}}{\Gamma\{{\downarrow}{\uparrow}(P\overset{+}{\wedge}Q),\langle{\downarrow}{\uparrow}Q\rangle\}}\qquad\vdots}{\Gamma\{{\downarrow}{\uparrow}(P\overset{+}{\wedge}Q),\langle{\downarrow}{\uparrow}P,{\downarrow}{\uparrow}Q\rangle\}}}{\Gamma\{{\downarrow}{\uparrow}(P\overset{+}{\wedge}Q)\}}$$

(where the omitted third premise derivation is the similar branch for $\langle{\downarrow}{\uparrow}P\rangle$), or it becomes

$$\mathsf{cut}^{\langle\rangle}\dfrac{\mathsf{weak}\,\dfrac{\Gamma\{{\downarrow}N,{\downarrow}M\}}{\Gamma\{{\downarrow}(N\overset{-}{\wedge}M),{\downarrow}N,{\downarrow}M\}}\qquad \mathsf{spl}^{\langle\rangle}\dfrac{\downarrow_\mathsf{R}^{\langle\rangle}\dfrac{\left\{\downarrow_\mathsf{L}^{\langle\rangle}\dfrac{\mathsf{sid}^{\langle\rangle}\dfrac{}{\Gamma^*\{{\downarrow}(N\overset{-}{\wedge}M),\langle\Delta_N\rangle,\Delta_N\}}}{\Gamma^*\{{\downarrow}(N\overset{-}{\wedge}M),\Delta_N\}}\right\}_{\Delta_N\in N}}{\Gamma\{{\downarrow}(N\overset{-}{\wedge}M),\langle{\downarrow}N\rangle\}}\qquad\vdots}{\Gamma\{{\downarrow}(N\overset{-}{\wedge}M),\langle{\downarrow}N,{\downarrow}M\rangle\}}}{\Gamma\{{\downarrow}(N\overset{-}{\wedge}M)\}}$$

(where the omitted third premise derivation is the similar branch for $\langle{\downarrow}M\rangle$). Finally,

$$\Diamond_{\mathsf{Rd}}\,\dfrac{\Lambda\{[A^\circ]\}}{\Lambda\{\Diamond A^\circ\}}\quad\text{becomes}\quad \mathsf{cut}^{\langle\rangle}\dfrac{\Lambda\{[{\uparrow}P]\}\qquad \mathsf{d}^{\langle\rangle}\dfrac{\left\{\uparrow_\mathsf{L}^{\langle\rangle}\dfrac{\uparrow_\mathsf{R}^{\langle\rangle}\dfrac{\mathsf{k}^{\langle\rangle}\dfrac{\mathsf{sid}^{\langle\rangle}\dfrac{}{\Lambda\{{\uparrow}\Diamond P,[\langle\Delta_P\rangle,\Delta_P]\}}}{\Lambda\{{\uparrow}\Diamond P,\langle[\Delta_P]\rangle,[\Delta_P]\}}}{\Lambda\{{\uparrow}\Diamond P,[\Delta_P]\}}}{\Lambda\{{\uparrow}\Diamond P,[\langle{\uparrow}P\rangle]\}}\right\}_{\Delta_P\in P}}{\Lambda\{{\uparrow}\Diamond P,\langle[{\uparrow}P]\rangle\}}}{\Lambda\{{\uparrow}\Diamond P\}}$$

The lemma then follows by replacing in the proof of P° (or N°) in NIK+X each instance of a rule by the corresponding derivation in SyNIK+X$^{\langle\rangle}$+cut, which builds a proof of ${\uparrow}P$ (or ${\uparrow}{\downarrow}N$ resp.) in SyNIK+X$^{\langle\rangle}$+cut. ◀

We can now summarize the results of this paper in the following theorem:

▶ **Theorem 6.9.** *Let* $\mathsf{X} \subseteq \{\mathsf{t},\mathsf{d},4,\mathsf{b},5\}$ *be 45-closed and let* Γ *be a neutral sequent. The following are equivalent:*

1. $\mathrm{fm}(\lfloor\Gamma\rfloor)$ *is* X*-valid.*
2. Γ *is provable in* FoNIK+X$^\mathsf{f}$.
3. Γ *is provable in* SyNIK+X$^{\langle\rangle}$.
4. Γ *is provable in* SyNIK+X$^{\langle\rangle}$+cut.

Proof. 4 → 3 is just Theorem 6.6; 3 → 2 follows from Lemma 5.4; 2 → 1 follows from Lemma 4.6; and finally 1 → 4 follows from Lemma 6.8 with the use of Theorem 3.8. Observe that the proof of Lemma 6.8 applies to unpolarized sequents and neutral sequents in general, and not just output formulas and right-neutral formulas. ◀

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