Focused Nested Sequents

Kaustuv Chaudhuri, Sonia Marin, and Lutz Straßburger

Inria & LIX/École polytechnique, France

Focusing is a general technique for transforming a sequent proof system into one with a syntactic separation of non-deterministic choices without sacrificing completeness. This discipline not only improves proof search, but also has the representational benefit of distilling sequent proofs into synthetic normal forms. A focused proof can be seen as an alternation of *negative* phases, where invertible rules are applied (bottom-up) eagerly, and *positive* phases, where applications of the other rules are confined and controlled.

The full theory of focusing was initially developed for the sequent calculus for linear logic [1], but it has since been extended to a wide variety of logics [5, 7, 8] and proof systems [4, 2]. We show how to apply the focusing technique to nested sequent calculi, a generalization of ordinary sequent calculi to tree-like instead of list-like structures. As nested sequent systems can be built *modularly* for every modal logic in the $\mathsf{S5}$ cube [3], this work improves the reach of focusing to the modal logics based on the axioms d, t, b, $\mathsf{4}$ and $\mathsf{5}$.

For simplicity we use a polarized syntax [6] consisting of two classes of positive and negative formulas and a pair of *shift* connectives to move back and forth between the classes (in particular, we interpret $\Diamond$ as positive and $\Box$ as negative):

$$\begin{array}{lrl} \text{positive:} & P, Q, \ldots & ::= a \mid P \mathbin{\overset{+}{\wedge}} Q \mid P \mathbin{\overset{+}{\vee}} Q \mid \Diamond P \mid {\downarrow} N \\ \text{negative:} & N, M, \ldots & ::= \bar{a} \mid N \mathbin{\bar{\vee}} M \mid N \mathbin{\bar{\wedge}} M \mid \Box N \mid {\uparrow} P \end{array}$$

In Figure 1, we present KNF the *focused* version of the nested sequent system for modal logic K (that we call KN) [3]. The rules for negative formulas are exactly the same as in the unfocused system, while the rules for positive formulas can only be applied if the principal formula is in focus (indicated by $\langle . \rangle$).

For any logic $\mathcal{L}$ in the $S5$-cube, there exists a *45-closed* set $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$, such that the system $\mathsf{KN} + \mathsf{X}^\diamond$ is sound and complete wrt. $\mathcal{L}$ [3]. In the second row of Figure 1 are the rules allowing to modularly extend KNF as $\mathsf{KNF} + \mathsf{X}^\diamond$, in a similar way. The soundness of the focused system is as usual straightforward.

Definition 1 (Depolarization). *If A is a polarized formula, then we write $\lfloor A \rfloor$ for the unpolarized formula obtained from A by erasing the shifts $\uparrow$ and $\downarrow$, collapsing $\overset{+}{\wedge}$ and $\bar{\wedge}$ (as well as $\overset{+}{\vee}$ and $\bar{\vee}$) into the classical $\wedge$ (and $\vee$ respectively).*

Theorem 1 (Soundness). *If a formula A is provable in $\mathsf{KNF} + \mathsf{X}^\diamond$, then $\lfloor A \rfloor$ is a theorem of $\mathsf{KN} + \mathsf{X}^\diamond$.*

Our first contribution is a purely *internal* proof of cut-elimination for the focused nested calculus, given in terms of a traditional rewriting procedure to eliminate cuts (which shows that our system is compositional and suitably continuous).

Theorem 2 (Cut-elimination). *Let $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ be 45-closed If a sequent Γ is provable in $\mathsf{KNF} + \mathsf{X}^\diamond + \{\mathsf{cut}_1, \mathsf{cut}_2, \mathsf{cut}_3\}$ (Figure 2), then it is also provable in $\mathsf{KNF} + \mathsf{X}^\diamond$.*

$$\mathsf{sto}\ \frac{\Gamma\{P\}}{\Gamma\{\uparrow P\}} \quad \mathsf{rel}\ \frac{\Gamma\{N\}}{\Gamma\{\langle\downarrow N\rangle\}} \quad \mathsf{dec}\ \frac{\Gamma\{P, \langle P\rangle\}}{\Gamma\{P\}}$$

$$\bar{\wedge}\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \mathbin{\bar{\wedge}} B\}} \quad \bar{\vee}\ \frac{\Gamma\{A, B\}}{\Gamma\{A \mathbin{\bar{\vee}} B\}} \quad \Box\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}}$$

$$\mathsf{id}\ \frac{}{\Gamma\{\bar{a}, \langle a\rangle\}} \quad \overset{+}{\wedge}\ \frac{\Gamma\{\langle A\rangle\} \quad \Gamma\{\langle B\rangle\}}{\Gamma\{\langle A \mathbin{\overset{+}{\wedge}} B\rangle\}} \quad \overset{+}{\vee}_1\ \frac{\Gamma\{\langle A\rangle\}}{\Gamma\{\langle A \mathbin{\overset{+}{\vee}} B\rangle\}} \quad \overset{+}{\vee}_2\ \frac{\Gamma\{\langle B\rangle\}}{\Gamma\{\langle A \mathbin{\overset{+}{\vee}} B\rangle\}} \quad \mathsf{k}^{\diamond}\ \frac{\Gamma\{[\langle A\rangle, \Delta]\}}{\Gamma\{\langle \diamond A\rangle, [\Delta]\}}$$

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$$\mathsf{d}^{\diamond}\ \frac{\Gamma\{[\langle A\rangle]\}}{\Gamma\{\langle \diamond A\rangle\}} \quad \mathsf{t}^{\diamond}\ \frac{\Gamma\{\langle A\rangle\}}{\Gamma\{\langle \diamond A\rangle\}} \quad \mathsf{b}^{\diamond}\ \frac{\Gamma\{[\Delta], \langle A\rangle\}}{\Gamma\{[\Delta, \langle \diamond A\rangle]\}}$$

$$\mathsf{4}^{\diamond}\ \frac{\Gamma\{[\langle \diamond A\rangle, \Delta]\}}{\Gamma\{\langle \diamond A\rangle, [\Delta]\}} \quad \mathsf{5}^{\diamond}\ \frac{\Gamma\{\emptyset\}\{\langle \diamond A\rangle\}}{\Gamma\{\langle \diamond A\rangle\}\{\emptyset\}}\ \mathrm{dp}(\Gamma\{\ \ \}\{\emptyset\}) \geq 1$$

Fig. 1. System $\mathsf{KNF} + \mathsf{X}^{\diamond}$.

$$\mathsf{cut}_1\ \frac{\Gamma\{P\} \quad \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}} \qquad \mathsf{cut}_2\ \frac{\Gamma\{\langle P\rangle\} \quad \Gamma\{\bar{P}\}}{\Gamma\{\emptyset\}} \qquad \mathsf{cut}_3\ \frac{\Gamma\{\langle Q\rangle\}\{P\} \quad \Gamma\{\emptyset\}\{\bar{P}\}}{\Gamma\{\langle Q\rangle\}\{\emptyset\}}$$

Fig. 2. The various cut-rules in KNF

Our second contribution is a proof of completeness of the focused system with respect to the non-focused system (and hence to the Kripke semantics) by showing that the focused system admits the rules of the non-focused system.

Theorem 3 (Completeness). *Let $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ be 45-closed For any A, if $\lfloor A \rfloor$ is provable in $\mathsf{KN} + \mathsf{X}^{\diamond}$, then A is provable in $\mathsf{KNF} + \mathsf{X}^{\diamond}$.*

References

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