Efficient multivariate low-degree tests
via interactive oracle proofs of proximity

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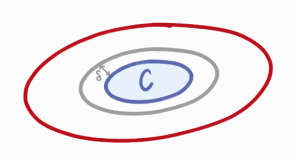
Joint work with Daniel Augot and Jade Nardi

WCC 2022
March 8, 2022

Topic of this talk

Proximity testing for a code

Given a (linear) code $C \subset \mathbb{F}^n$ and oracle access to $f \in \mathbb{F}^n$, algorithmically distinguish between $\begin{cases} f \in C \\ f \text{ is } \delta\text{-far from } C \end{cases}$ with $q = o(n)$ queries to f.



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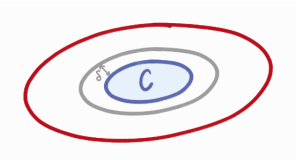

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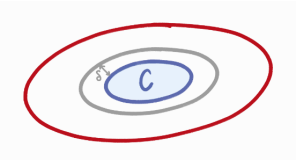

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- **Tensor product of RS codes:**

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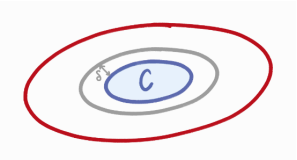

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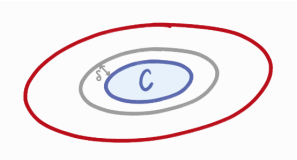

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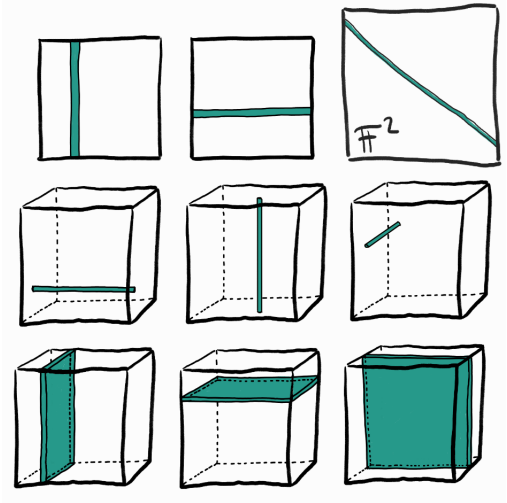
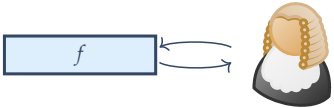
Proximity tests for polynomial codes ⟷ **Low-degree tests**

Multivariate polynomial codes are locally testable





Axis-parallel tests
→ **Test individual degree**

Random line tests
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(Require evaluation domain = $\mathbb{F}^m$)

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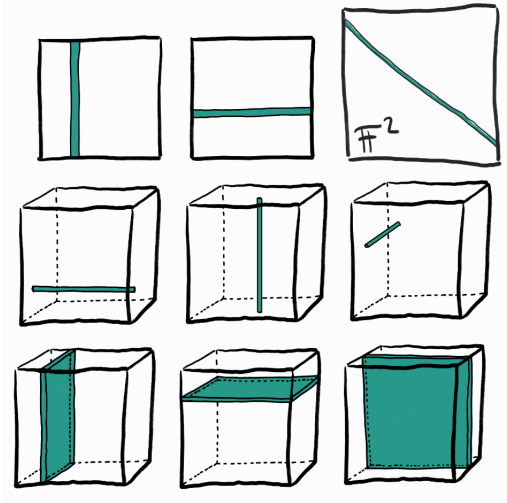

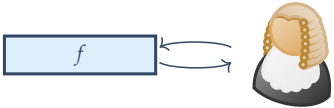

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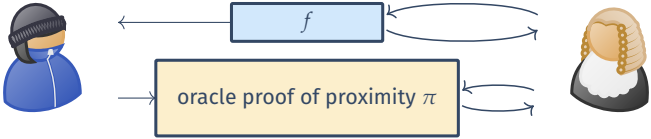
Oracle access to f only
→ **At least d queries**

Testing proximity in the PCP model

Probabilistically Checkable Proof of Proximity (PCPP):



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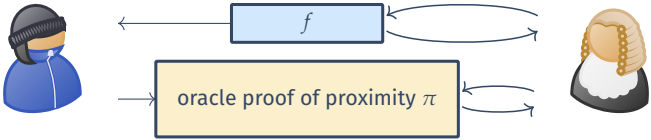

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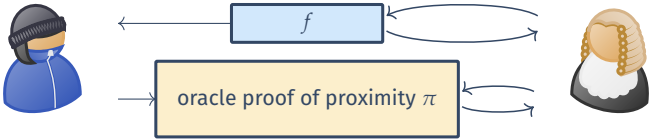
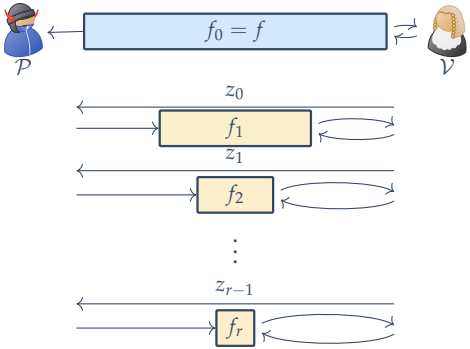


- Ask a prover to provide an **auxiliary proof** π
- Enable sublinear tests for **non-locally testable codes** (e.g. Reed-Solomon codes)
- Prover must compute in advance the answers to all possible queries ⟶ **impractical**

Interactive Oracle Proof of Proximity



DEF IOPP for a Code

An **Interactive Oracle Proof of Proximity (IOPP)** $(\mathcal{P}, \mathcal{V})$ for C with soundness error $s : (0,1] \to [0,1]$ satisfies:

Completeness

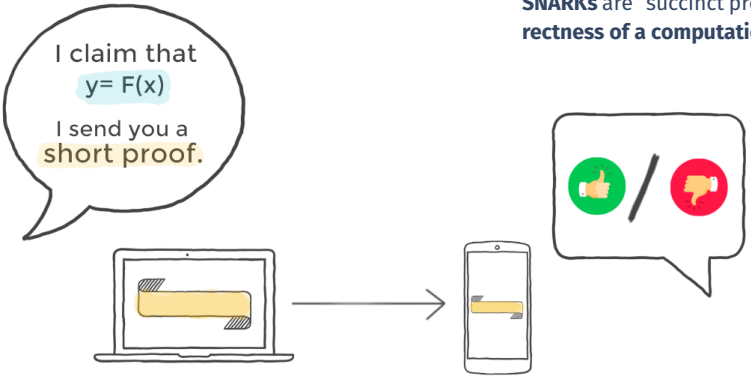
If $f \in C$, then $\exists \mathcal{P} \Pr[\langle \mathcal{P}(f), \mathcal{V}^f \rangle = 1] = 1$.

Soundness

If f is δ-far from C,
Then, for all unbounded $\tilde{\mathcal{P}}$, $\Pr[\langle \tilde{\mathcal{P}}, \mathcal{V}^f \rangle = 1] \leq s(\delta)$.

Motivation: (ZK-)SNARKs

SNARKs are “succinct proofs” that enable to check **correctness of a computation**, with/without **ZK**.



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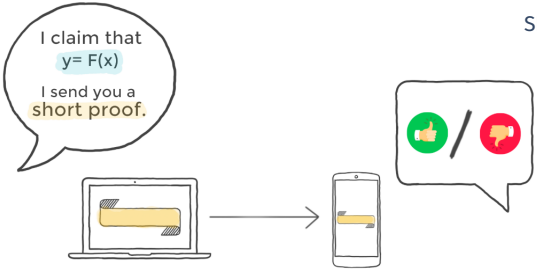

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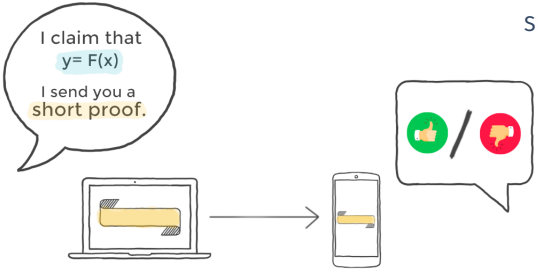

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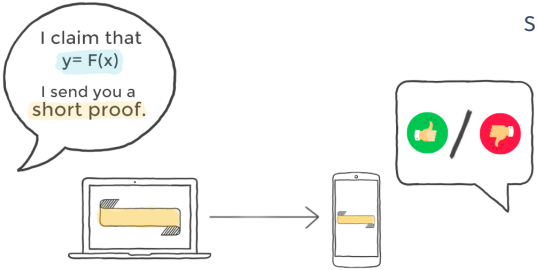

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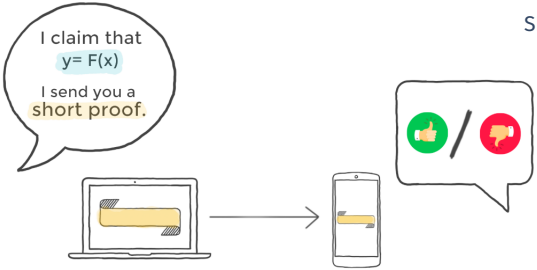

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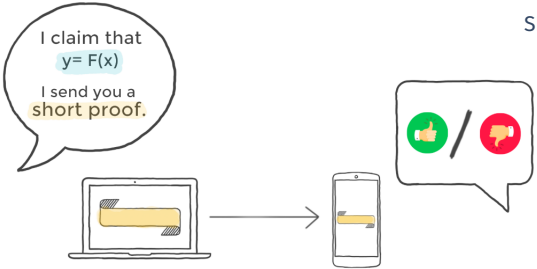

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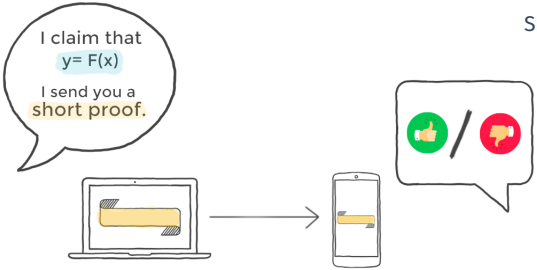

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- **Key component = IOP of Proximity** for codes, with
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- Size of SNARK:
  - linear in query complexity of IOP
  - polylog in proof length of IOP

Our results

IOPPs for testing proximity to multivariate polynomial codes.

Length is N, number of variables is m.

Regarding SNARKs applications, constant rate codes ⟶ shorter proofs ($m =$ constant)

	Type	Prover	Verifier	Query	Length	Rounds
[BBHR18]	RS $(m = 1)$	$< 11N$	$< 11 \log N$	$< 2 \log N$	$< N$	$< \log N$
This work	RS$^{\otimes m}$	$< 11N$	$< 11 \log N$	$< 2 \log N$	$< N$	$< \log N$
This work	RM	$< (2m+7)N$	$< 2^m \left(\frac{5}{4} + \frac{7}{m}\right) \log N$	$< \frac{2^m}{m} \log N$	$< \frac{N}{2^m - 1}$	$< \frac{\log N}{m}$

(Complexities counted in $\mathbb{F}$-ops and field elements)

Reed-Solomon IOP of Proximity

Reed-Solomon IOPP

DEF RS code

Given $L \subset \mathbb{F}$, $d < |L|$,
$\mathsf{RS}[L, d] = \left\{ f_{|L} : L \to \mathbb{F} \mid f \in \mathbb{F}[X], \deg f < d \right\}$

The **FRI** protocol is a "**F**ast **R**eed-Solomon **I**OPP" [BBHR18]

Setting:

- L is coset of mult. or add. subgroup of $\mathbb{F}$,
  $|L| =$ power of 2
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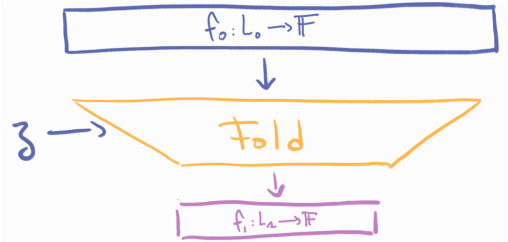

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Recursively halve the size of the problem via "random folding".

Halve the size of the problem

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Given **arbitrary** function $f : L \to \mathbb{F}$,

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- For $z \in \mathbb{F}$, define **Fold** $[f, z] : q(L) \to \mathbb{F}$ by

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Linear system $\implies$ $g_0(y) = \frac{f(x)+f(-x)}{2}$ and $g_1(y) = \frac{f(x)-f(-x)}{2x}$.

The 3 key properties

Key properties of folding operators

1. Completeness:

$$f \in \mathsf{RS}[L, d] \implies \textbf{Fold}\,[f, z] \in \mathsf{RS}[q(L), d/2] \qquad \text{for all } z.$$

2. Local computability:

Each entry of **Fold** $[f, z]$ depends on $l = 2$ entries of f.

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3. Distance preservation:

f is δ-far from $\mathsf{RS}[L, d] \implies$ **Fold** $[f, z]$ is δ'-far from $\mathsf{RS}[q(L), d/2]$ w.h.p over z $(\delta' \approx \delta)$

IOPP: COMMIT phase

$\mathsf{RS}[L, d = 2^r]$

Honest prover computes:

$\mathcal{P}$

$\mathcal{V}$

$f_0 = f : L \to \mathbb{F}$

$z_0 \xleftarrow{\$} \mathbb{F}$

$f_1 = \textbf{Fold}\,[f_0, z_0]$

f_1

z_1

$f_2 = \textbf{Fold}\,[f_1, z_1]$

f_2

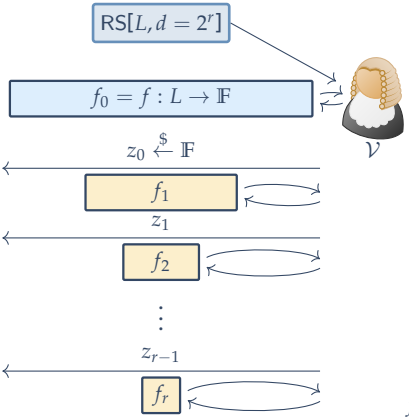
$\vdots$

z_{r-1}

$f_r = \textbf{Fold}\,[f_{r-1}, z_{r-1}] \equiv c \in \mathbb{F}$

f_r

IOPP: QUERY phase



Global consistency test:

Sample $s \in L$ and check

$f_1(s^2) \stackrel{?}{=} \textbf{Fold}\,[f_0, z_0]\,(s^2)$

$f_2(s^4) \stackrel{?}{=} \textbf{Fold}\,[f_1, z_1]\,(s^4)$

$\vdots$

$f_r(s^{2^r}) \stackrel{?}{=} \textbf{Fold}\,[f_{r-1}, z_{r-1}]\,(s^{2^r})$

Final test: $f_r \stackrel{?}{=} c \in \mathbb{F}$

IOPP

$\mathsf{RS}[L, d = 2^r]$

Honest prover computes:

$\mathcal{P}$

$f_0 = f : L \to \mathbb{F}$

$\mathcal{V}$

Global consistency test:

$z_0 \xleftarrow{\$} \mathbb{F}$

$f_1 = \textbf{Fold}\,[f_0, z_0]$ — f_1 — Sample $s \in L$ and check

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z_1

$f_2 = \textbf{Fold}\,[f_1, z_1]$ — f_2 — $f_2(s^4) \stackrel{?}{=} \textbf{Fold}\,[f_1, z_1]\,(s^4)$

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z_{r-1}

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Final test: $f_r \stackrel{?}{=} c \in \mathbb{F}$

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IOPP

$\mathsf{RS}[L, d = 2^r]$

Honest prover computes:

$\mathcal{P}$

$f_0 = f : L \to \mathbb{F}$

$\mathcal{V}$

Global consistency test:

$z_0 \xleftarrow{\$} \mathbb{F}$

$f_1 = \textbf{Fold}\,[f_0, z_0]$ — f_1 — Sample $s \in L$ and check

$f_1(s^2) \stackrel{?}{=} \textbf{Fold}\,[f_0, z_0]\,(s^2)$

z_1

$f_2 = \textbf{Fold}\,[f_1, z_1]$ — f_2 — $f_2(s^4) \stackrel{?}{=} \textbf{Fold}\,[f_1, z_1]\,(s^4)$

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How to fold multivariate polynomials

Folding tensor product of RS codes

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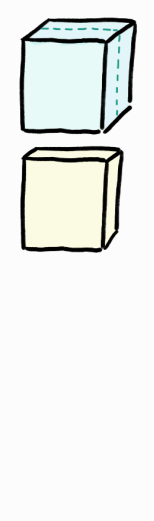
- $\mathsf{RS}^{\otimes m} \longrightarrow \mathsf{RS}^{\otimes m-1}$:



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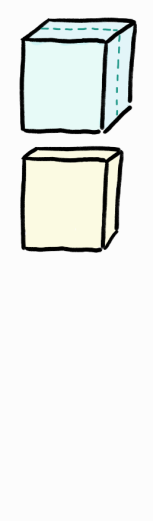

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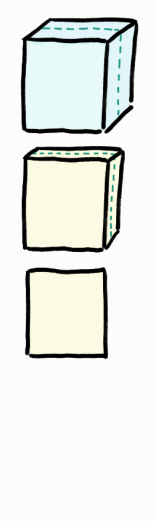

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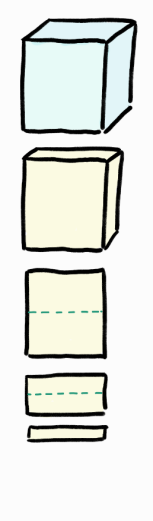

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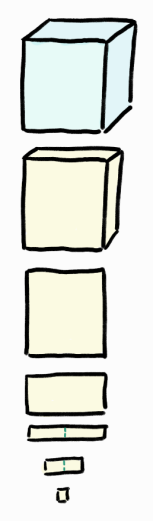

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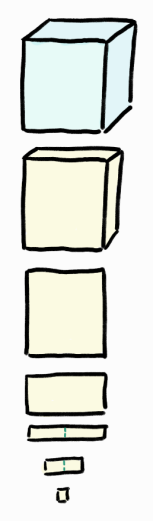

☑ **Completeness** ☑ **Local computability** ☑ **Distance preservation**

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In the total degree case, we fold along **every variable at the same time**.

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Subtlety: need to be careful about the distinct degree bounds on the $g_{\boldsymbol{u}}$'s.

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Conclusion

What we have:

- Inspired by FRI protocol for RS-IOPP,
  we give concrete IOPPs for $\mathsf{RS}^{\otimes m}$ and RM codes with **similar parameters**.

THEOREM [Augot-Bordage-Nardi'21]

$\mathsf{RS}[L, d]^{\otimes m}$ has an IOPP $(\mathcal{P}, \mathcal{V})$ satisfying

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