INVARIANTS FOR MATRIX CODES: A TENSOR ALGEBRA APPROACH

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OVERVIEW

- INTRODUCTION
  - WHAT IS COMPLEXITY?
  - 3-TENSORS
  - APPLICATIONS OF TENSOR DECOMPOSITION
- TENSOR RANK OF $\mathbb{F}_{q^m}$-LINEAR MATRIX CODES
- INVARIANTS FOR MATRIX CODES

WHAT IS COMPLEXITY ?

Definition

The **complexity** of a *problem* is the cost of the optimal procedure among all the ones that solve the *problem* and fit into a given model of computation.

- It is allowed to freely use the *intermediate results* once they are computed.

- A *computation* is said to be **finished** if the quantities that the computation is supposed to compute are among the *intermediate results*.

WHAT IS COMPLEXITY ?

- The cost of a *computation* that solves a problem is an **upper bound** on the complexity of that problem with respect to the given model.

- **Lower bounds** can be often obtain by establishing relations between the complexity of the problem and the invariants of the appropriate structure (algebraic, topological, geometric or combinatorial).

- We are interested in the so-called **nonscalar model** where additions, subtractions and scalar multiplications are free of charge. The (**nonscalar**) **cost** of an algorithm is therefore the number of multiplications and divisions needed to compute the result.

AN EXAMPLE: MULTIPLICATION OF 2×2 MATRICES

Let A, B be 2×2 following matrices

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}.$$

The standard algorithm returns the matrix $C = AB$ by computing the following intermediate results:

$$\begin{aligned} c_1 &= a_1 b_1 + a_2 b_3, \qquad & c_2 &= a_1 b_2 + a_2 b_4, \\ c_3 &= a_3 b_1 + a_4 b_3, \qquad & c_4 &= a_3 b_2 + a_4 b_4. \end{aligned}$$

It requires 8 **multiplications** and 4 **additions**. Therefore, an **upper bound** for the complexity (in the nonscalar model) is 8.

AN EXAMPLE: MULTIPLICATION OF 2×2 MATRICES

We can compute $C = AB$ using Strassen's algorithm, which gives

$$c_1 = S_1 + S_4 - S_5 + S_7, \qquad c_2 = S_2 + S_4, \qquad c_3 = S_3 + S_5, \qquad c_4 = S_1 + S_3 - S_2 + S_6$$

where the S_i's are the intermediate steps

$$\begin{aligned} S_1 &= (a_1 + a_4)(b_1 + b_4), & S_2 &= (a_3 + a_4)b_1, & S_3 &= a_1(b_3 - b_4), \\ S_4 &= a_4(b_3 - b_1), & S_5 &= (a_1 + a_2)b_4, & S_6 &= (a_3 - a_1)(b_1 + b_2), \\ S_7 &= (a_2 - a_4)(b_3 + b_4). \end{aligned}$$

It requires 7 **multiplications** and 18 **additions**.

AN EXAMPLE: MULTIPLICATION OF 2×2 MATRICES

Algorithm	# multiplication	# additions
standard	8	4
Strassen's	7	18

Remark

The complexity of multiplying 2×2 matrices (in the nonscalar model) is 7. The upper-bound is given by Strassen (1969), the lower bound was proved by Winograd (1971).

Let A, B be vector spaces over the same field $\mathbb{K}$ and denote by A^* the **dual vector space** of A, i.e. $A^* := \{f : A \longrightarrow \mathbb{K} \mid f \text{ linear}\}$. For $\alpha \in A^*$ and $b \in B$, one can define a *rank one* linear map

$$\alpha \otimes b : A \longrightarrow B : a \longmapsto \alpha(a)b.$$

Definition

The **rank** $\tau(f)$ of a linear map $f : A \longrightarrow B$ is the smallest integer R such that there exist $\alpha_1, \ldots, \alpha_R \in A^*$ and $b_1, \ldots, b_R \in B$ such that

$$f = \sum_{i=1}^{R} \alpha_i \otimes b_i.$$

BILINEAR MAPS

Let A, B, C be vector spaces over the same field $\mathbb{K}$. For $\alpha \in A^*$, $\beta \in B^*$ and $c \in C$, one can define a *rank one* bilinear map

$$\alpha \otimes \beta \otimes c : A \times B \longrightarrow C : (a, b) \longmapsto \alpha(a)\beta(b)c.$$

Definition

The **rank** $\tau(T)$ of a bilinear map $T : A \times B \longrightarrow C$ is the smallest integer R such that there exist $\alpha_1, \ldots, \alpha_R \in A^*$, $\beta_1, \ldots, \beta_R \in B^*$ and $c_1, \ldots, c_R \in C$ such that

$$T = \sum_{i=1}^{R} \alpha_i \otimes \beta_i \otimes c_i.$$

BILINEAR MAPS AND COMPLEXITY

- If a bilinear map T has rank R then T can be *executed* by performing R multiplications (and $\mathcal{O}(R)$ additions).
- The rank of a bilinear map gives a measure of its complexity.

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Example

Matrix multiplication of $n \times n$ matrices is a bilinear map:

$$M_{n,n,n} : \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times n} \longrightarrow \mathbb{K}^{n\times n}.$$

We observed that $R(M_{2,2,2}) = 7$ and it is known that $19 \leq R(M_{3,3,3}) \leq 23$.

3 - TENSORS

We assume n, m, k to be integers.

Definition

A 3-**tensor** is an element of $\mathbb{K}^k \otimes \mathbb{K}^n \otimes \mathbb{K}^m$.

If $\{a_1, \ldots, a_k\}, \{b_1, \ldots, b_n\}, \{c_1, \ldots, c_m\}$ are bases of $\mathbb{K}^k, \mathbb{K}^n, \mathbb{K}^m$, respectively, then a basis for $\mathbb{K}^k \otimes \mathbb{K}^n \otimes \mathbb{K}^m$ is

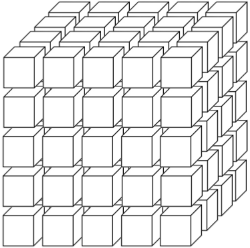
$$\{a_i \otimes b_j \otimes c_\ell : 1 \leq i \leq k, 1 \leq j \leq n, 1 \leq \ell \leq m\}.$$

In particular we have $\dim(\mathbb{K}^k \otimes \mathbb{K}^n \otimes \mathbb{K}^m) = \dim(\mathbb{K}^k)\dim(\mathbb{K}^n)\dim(\mathbb{K}^m) = knm$.

COORDINATE TENSORS

A tensor $X := \sum_r a_r \otimes b_r \otimes c_r$ can be represented as an array. That is as the map

$$X : \{1, \ldots, k\} \times \{1, \ldots, n\} \times \{1, \ldots, m\} \longrightarrow \mathbb{K}$$



given by $X = (X_{ij\ell} : 1 \leq i \leq k, 1 \leq j \leq n, 1 \leq \ell \leq m)$.

Therefore, X is related to the the 3-dimensional array

$$X_{ij\ell} = \sum_r a_{\ell r} b_{ir} c_{jr}.$$

where $a_r := (a_{\ell r} : 1 \leq \ell \leq k), b_r := (b_{ir} : 1 \leq i \leq n), c_r := (a_{jr} : 1 \leq j \leq m)$.

Remark

This representation of X is called **coordinate tensor** and allows to identify the space $\mathbb{K}^k \otimes \mathbb{K}^n \otimes \mathbb{K}^m$ with $\mathbb{K}^{k \times n \times m}$.

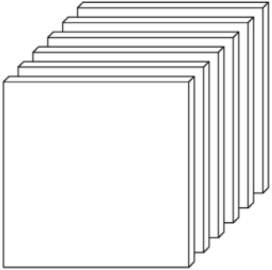
MATRIX REPRESENTATION

Consider the map $\mu : \mathbb{K}^k \times \mathbb{K}^{k\times n\times m} \longrightarrow \mathbb{K}^{k\times n\times m} : (v, X) \longmapsto \sum_r (v \cdot a_r) \otimes b_r \otimes c_r$, and notice that this map yields a 3-tensor of the form $\sum_r \lambda_r \otimes b_r \otimes c_r$, where $\lambda_r \in \mathbb{K}$, which can be identify as the 2-tensor $\sum_r \lambda_r b_r \otimes c_r$, since $\mathbb{K} \otimes \mathbb{K}^n$ and $\mathbb{K}^n$ are isomorphic.

As a consequence, we can identify the tensor X with the array of $n \times m$ matrices $X = (X_1 \mid \ldots \mid X_k)$, where

$$X_s := \mu(e_s, X) = \sum_r (a_r)_s b_r \otimes c_r$$

and e_s is the s-th element of the canonical basis for $\mathbb{K}^k$, for all $1 \leq s \leq k$.



3 - TENSORS

Let $X = (X_1 \mid \ldots \mid X_k) \in \mathbb{K}^{k \times n \times m}$ be a 3-tensor.

Definition

The **first slice space** $\mathrm{ss}_1(X)$ of X is defined as the span $\langle X_1, \ldots, X_k \rangle$ over $\mathbb{K}$. We say that $\mathrm{ss}_1(X)$ is **nondegenerate** if $\dim(\mathrm{ss}_1(X)) = k$.

Definition

X is said to be **simple** (or **rank one**) if there exist $a \in \mathbb{K}^k$, $b \in \mathbb{K}^n$ and $c \in \mathbb{K}^m$ such that $X = a \otimes b \otimes c$.

Definition

The **tensor rank** $\mathrm{trk}(X)$ of X is defined as the smallest R such that X can be expressed as sum of R simple tensors.

PERFECT BASE

Let $X = (X_1 \mid \ldots \mid X_k) \in \mathbb{K}^{k \times n \times m}$ be a 3-tensor.

Definition

Let $\mathcal{A} := \{A_1, \ldots, A_R\} \subseteq \mathbb{K}^{n \times m}$ be a set of R linearly independent rank-1 matrices. We say that $\mathcal{A}$ is a **perfect base** (or R-**base**) for the tensor X if

$$\mathrm{ss}_1(X) \leq \langle A_1, \ldots, A_R \rangle .$$

Lemma

The following are equivalent.

- $\mathrm{trk}(X) \leq R$.
- There exists an R-base for X.

AN EXAMPLE

Let $X \in \mathbb{F}_5^{2\times 2\times 2}$ be the 3-tensor defined as

$$X := \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & 1 \end{array}\right).$$

One can check that $\operatorname{trk}(X) = 3$ and a 3-base for X is given by

$$\mathcal{A} := \left\{ \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \right\}.$$

In particular, we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 \\ 3 & 1 \end{pmatrix} = 2\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}$$

EQUIVALENT 3-TENSORS

Let $X = (X_1 \mid \dots \mid X_k)$ and $Y = (Y_1 \mid \dots \mid Y_k)$ be 3-tensors in $\mathbb{K}^{k \times n \times m}$.

Definition

We say that X, Y are **equivalent** if there exist $P \in \mathrm{GL}_n(\mathbb{K})$ and $Q \in \mathrm{GL}_m(\mathbb{K})$ such that $\mathrm{ss}_1(X) = P\,\mathrm{ss}_1(Y)\,Q := \{P\,N\,Q : N \in \mathrm{ss}_1(Y)\}$.

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Remark

For any pair of matrices $P \in \mathrm{GL}_n(\mathbb{K})$ and $Q \in \mathrm{GL}_m(\mathbb{K})$, if $\mathcal{A}$ is a perfect base for X then $\{P\,A\,Q : A \in \mathcal{A}\}$ is a perfect base for the 3-tensor $P\,X\,Q$.

APPLICATIONS OF TENSOR DECOMPOSITION

- Cumulants
  (Statistics)
- Fluorescence spectroscopy
  (Chemistry)
- Interpretation of MRI
  (Medicine)
- Blind source separation
  (e.g. Cocktail Party Problem)
  (Digital Signal Processing)
- Storage and Encoding
  (Coding Theory)

$$K(t) = \sum_{i=0}^{\infty} \kappa_n \frac{t^n}{n!} = \mu t + \sigma^2 \frac{t^2}{2} + \cdots$$

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Low tensor rank 3-tensors perform well in terms of storage and encoding complexity!

ISSUES IN TENSOR DECOMPOSITION

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Noise: in order to talk about noise in data, we must have a distance function. In some applications, these functions come from science, in other case they are chosen by convenience. For example, in signal processing, assuming that the noise has a certain behaviour (iid or Gaussian) can determine a distance function.

TENSOR RANK OF $\mathbb{F}_{q^m}$-LINEAR CODES

In the following, we assume $n \leq m$ without loss of generality.

Definition

A (**matrix rank-metric**) **code** is a subspace $\mathcal{C} \leq \mathbb{F}_q^{n\times m}$. The **minimum** (**rank**) **distance** of a non-zero code $\mathcal{C}$ is $d(\mathcal{C}) := \min(\{\mathrm{rk}(c) : c \in \mathcal{C}, c \neq 0\})$ and for $\mathcal{C} := \{0\}$, we define $d(\mathcal{C})$ to be $n+1$. The **maximum-rank** of $\mathcal{C}$ is defined as $\mathrm{maxrk}(\mathcal{C}) = \max\{\mathrm{rk}(c) : c \in C\}$.

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Proposition (Kruskal - 1977)

We have that $\mathrm{trk}(\mathcal{C}) \geq \dim_{\mathbb{F}_q}(\mathcal{C}) + d(\mathcal{C}) - 1$.

Codes meeting this bound are called **MTR** (**Minimal Tensor Rank**).

$\mathbb{F}_{q^m}$-LINEAR RANK-METRIC CODES

Let $\Gamma := \{\gamma_1, \dots, \gamma_m\}$ be a basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$ and $v \in \mathbb{F}_{q^m}^n$. We define by $\Gamma(v) \in \mathbb{F}_q^{n\times m}$ the vector defined by

$$v_i = \sum_{j=1}^{m} \Gamma(v)_{i,j}\, \gamma_j.$$

The map $v \mapsto \Gamma(v)$ is an $\mathbb{F}_q$-isomorphism. Moreover, for a subspace V of $\mathbb{F}_{q^m}^n$, we define $\Gamma(V) := \{\Gamma(v) : v \in V\}$.

Definition

A **vector (rank-metric) code** is a subspace $C \leq \mathbb{F}_{q^m}^n$. The **minimum distance** $d(C)$ of C is the minimum distance of $\Gamma(C)$ for any choice of a basis Γ of $\mathbb{F}_{q^m}/\mathbb{F}_q$.

$\mathbb{F}_{q^m}$-LINEAR RANK-METRIC CODES

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A vector code C is MTR if $\mathrm{trk}(C) = \dim_{\mathbb{F}_q}(C) + d(C) - 1$.

Definition

Let $\beta_1, \dots, \beta_n$ be elements of $\mathbb{F}_{q^m}$ linearly independent over $\mathbb{F}_q$. We define the k-dimensional $\mathbb{F}_{q^m}$**-Delsarte-Gabidulin code** $\mathcal{G}_k(\beta_1, \dots, \beta_n)$ as

$$\mathcal{G}_k(\beta_1, \dots, \beta_n) := \{(f(\beta_1), \dots, f(\beta_n)) : f \in \mathcal{G}_k\},$$

where $\mathcal{G}_k := \{f_0 x + f_1 x^q + \cdots + f_{k-1} x^{q^{k-1}} : f_0, \dots, f_{k-1} \in \mathbb{F}_{q^m}\}$.

DELSARTE- GABIDULIN CODES

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Proposition (Sheekey - 2016)

Let $\beta_1, \ldots, \beta_n$ be elements of $\mathbb{F}_{q^m}$ linearly independent over $\mathbb{F}_q$. The dual of the code $\mathcal{G}_k(\beta_1, \ldots, \beta_n)$ is equivalent to $\mathcal{G}_{n-k,s}(\beta_1, \ldots, \beta_n)$.

AN EXAMPLE

Let α be a primitive element of $\mathbb{F}_{5^3}$ and let

$$\begin{aligned} C := \mathcal{G}_1\left(\alpha^4, \alpha^7\right) &= \left\{\left(f\left(\alpha^4\right), f\left(\alpha^7\right)\right) : f \in \{f_0\, x : f_0 \in \mathbb{F}_{5^3}\}\right\} \\ &= \left\{f_0\left(\alpha^4, \alpha^7\right) : f_0 \in \mathbb{F}_{5^3}\right\} = \left\langle\left(\alpha^4, \alpha^7\right)\right\rangle_{\mathbb{F}_5}. \end{aligned}$$

Let $\Gamma := \{1, \alpha, \alpha^2\}$ be a $\mathbb{F}_5$-basis of $\mathbb{F}_{5^3}$, $N := \Gamma\left(\left(\alpha^4, \alpha^7\right)\right)$ and M the companion matrix of the minimal polynomial of α, i.e.

$$N := \begin{pmatrix} 0 & 2 & 2 \\ 3 & 2 & 3 \end{pmatrix}, \qquad M := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 2 & 0 \end{pmatrix}.$$

One can check that

$$\Gamma(C) = \langle N, N\,M, N\,M^2\rangle_{\mathbb{F}_5} = \left\langle \begin{pmatrix} 0 & 2 & 2 \\ 3 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 4 & 2 \\ 1 & 4 & 2 \end{pmatrix}, \begin{pmatrix} 4 & 3 & 4 \\ 4 & 0 & 4 \end{pmatrix} \right\rangle_{\mathbb{F}_5}.$$

Proposition (Byrne, Neri, Ravagnani, Sheekey - 2019)

Let $q \geq m + n - 2$ and α be primitive element of $\mathbb{F}_{q^m}$. For any code $C \leq \mathbb{F}_{q^m}^n$ equivalent to $\mathcal{G}_1(1, \alpha, \ldots, \alpha^n)$ we have

$$\mathrm{trk}(C) = m + n - 1$$

and, in particular, C is MTR.

Algebras Having Linear Multiplicative Complexities, *C. M. Fiduccia, Y. Zalcstein*
Journal of the ACM (JACM), ACM, 1977.

Proposition (Byrne, C. - 2021)

Let $q \geq m+n-2$, $n \in \{2,3\}$ and α be primitive element of $\mathbb{F}_{q^m}$. We can **construct** a perfect base of cardinality $m+n-1$ for any code $C \leq \mathbb{F}_{q^m}^n$ equivalent to $\mathcal{G}_1(1, \alpha, \ldots, \alpha^n)$.

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Proposition (Byrne, C. - 2021)

Let $q \geq m$ and α be primitive element of $\mathbb{F}_{q^m}$. For any code $C \leq \mathbb{F}_{q^m}^n$ equivalent to $\mathcal{G}_1(1,\alpha,\dots,\alpha^n)^\perp$ we have

$$\mathrm{trk}(C) = mn - m + 1$$

and, in particular, C is MTR. Moreover, we can **construct** a perfect base of cardinality $mn-m+1$ for C.

AN EXAMPLE

Let α be a primitive element of $\mathbb{F}_{5^3}$ and let $C := \mathcal{G}_1(1, \alpha, \alpha^2) = \langle(1, \alpha, \alpha^2)\rangle_{\mathbb{F}_5}$. One can check that $\Gamma\left(C^{\perp}\right) \leq \mathbb{F}_5^{3\times 3}$ is the code of dimension 6 generated by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 3 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix}.$$

Moreover, we have that $\mathrm{trk}\left(C^{\perp}\right) = 7$ and a 7-base for $C^{\perp}$ is given by the following rank-1 matrices

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 3 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 2 & 1 \\ 2 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 4 & 4 & 4 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & 1 \\ 3 & 4 & 4 \end{pmatrix}.$$

In particular, the span over $\mathbb{F}_5$ of these rank-1 contains $C^{\perp}$ as subspace.

INVARIANTS FOR MATRIX CODES

PRELIMINARIES AND NOTATION

Definition

The **row-support** and the **column-support** of a code $\mathcal{C} \leq \mathbb{F}_q^{n\times m}$ are

$$\mathrm{rowsupp}(\mathcal{C}) = \sum_{c\in\mathcal{C}} \mathrm{rowsp}(c) \qquad \text{and} \qquad \mathrm{colsupp}(\mathcal{C}) = \sum_{c\in\mathcal{C}} \mathrm{colsp}(c),$$

where, for any $c \in \mathcal{C}$, $\mathrm{rowsp}(c)$ and $\mathrm{colsp}(c)$ denotes the row-space and the column-space of c respectively.

Definition

Let $V \leq \mathbb{F}_q^m$, $U \leq \mathbb{F}_q^n$ and $\mathcal{C} \leq \mathbb{F}_q^{n\times m}$ be a code. We define

$$\mathcal{C}[V] := \{c \in \mathcal{C} : \mathrm{rowsp}(c) \leq V\} \qquad \text{and} \qquad \mathcal{C}(U) := \{c \in \mathcal{C} : \mathrm{colsp}(c) \leq U\}.$$

DELSARTE-TYPE ANTICODES

Definition (Ravagnani - 2016)

Let $\mathcal{C} \leq \mathbb{F}_q^{n\times m}$ be a code. We say that $\mathcal{C}$ is a **Delsarte-type anticode** if

$$\dim_{\mathbb{F}_q}(\mathcal{C}) = m \cdot \mathrm{maxrk}(\mathcal{C}).$$

Theorem (Meshulam - 1985)

Let $\mathcal{C} \leq \mathbb{F}_q^{n\times m}$ be a code. We have that $\mathcal{C}$ is a Delsarte-type anticode if and only if one of the following condition holds.

- $n < m$ and there exists $U \leq \mathbb{F}_q^{n}$ such that $\mathcal{C} = \mathbb{F}_q^{n\times m}(U)$.
- $n = m$ and there exists $U \leq \mathbb{F}_q^{n}$ such that $\mathcal{C} = \mathbb{F}_q^{n\times m}(U)$ or $\mathcal{C} = \mathbb{F}_q^{n\times m}[U]$.

TENSOR REPRESENTATION OF MATRICES

Consider the following map.

$$
\begin{aligned}
\varphi : \quad \mathbb{F}_q^n \otimes \mathbb{F}_q^m \quad &\longrightarrow \quad \mathbb{F}_q^{n\times m} \\
\sum_{i=1}^{R} \lambda_i\, u_i \otimes v_i \quad &\longmapsto \quad \sum_{i=1}^{R} \lambda_i \underbrace{\begin{pmatrix} u_1v_1 & u_1v_2 & \cdots & u_1v_m \\ u_2v_1 & u_2v_2 & \cdots & u_2v_m \\ \vdots & \vdots & & \vdots \\ u_nv_1 & u_nv_2 & \cdots & u_nv_m \end{pmatrix}}_{\text{rank-1 matrix}}
\end{aligned}
$$

Remark

One can easily check that the map φ is an isomorphism. Therefore, we can identify the spaces $\mathbb{F}_q^n \otimes \mathbb{F}_q^m$ and $\mathbb{F}_q^{n\times m}$.

DELSARTE-TYPE ANTICODES

Observe that for any $U \leq \mathbb{F}_q^n$ we have

$$\mathbb{F}_q^{n\times m}(U) = \left\{ \sum_{i=1}^{R} \lambda_i \, u_1 \otimes v_i : u_1, \dots, u_R \in U \text{ and } v_1, \dots, v_R \in \mathbb{F}_q^m \right\} = U \otimes \mathbb{F}_q^m.$$

Analogously, for $V \in \mathbb{F}_q^m$ we have $\mathbb{F}_q^{n\times m}[V] = \mathbb{F}_q^n \otimes V$.

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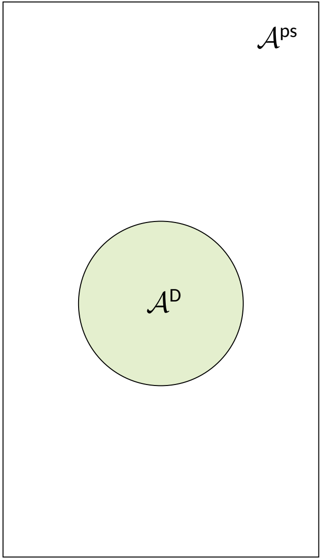
A TENSOR ALGEBRA APPROACH TO ANTICODES



$$\mathcal{A}^{\mathrm{ps}} := \{A \leq \mathbb{F}_q^{n\times m} : A \text{ is perfect}\},$$

$$\mathcal{A}^{\mathrm{D}} := \begin{cases} \{U \otimes \mathbb{F}_q^m : U \leq \mathbb{F}_q^n\} & \text{if } n < m, \\ \{U \otimes \mathbb{F}_q^n : U \leq \mathbb{F}_q^n\} \cup \{\mathbb{F}_q^n \otimes U : U \leq \mathbb{F}_q^n\} & \text{if } n = m, \end{cases}$$

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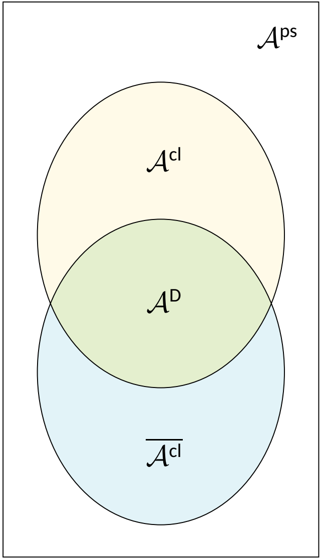

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$$\mathcal{A}^{\mathrm{cl}} := \{U \otimes V : U \leq \mathbb{F}_q^n \text{ and } V \leq \mathbb{F}_q^m\},$$

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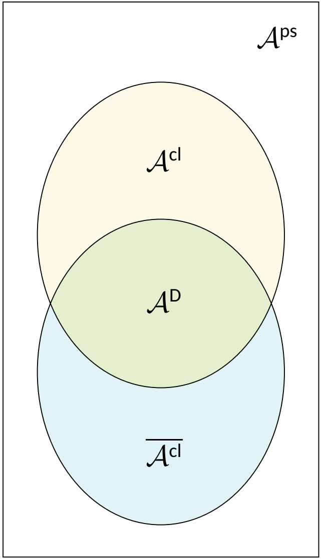

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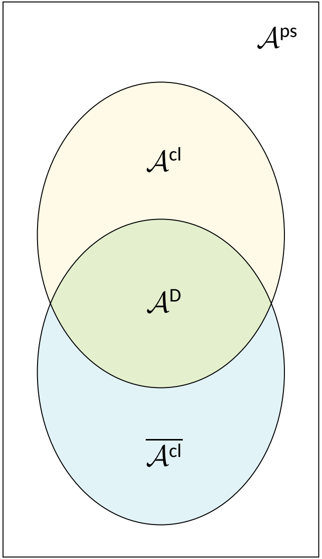
Closure-type anticodes

A TENSOR ALGEBRA APPROACH TO ANTICODES



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Closure-type anticodes

$$\boxed{A \in \mathcal{A}^{\mathrm{cl}} \iff A^{\perp} \in \overline{\mathcal{A}^{\mathrm{cl}}}}$$

INVARIANTS FOR MATRIX CODES

Definition

Let $\mathcal{C} \leq \mathbb{F}_q^{n\times m}$ be a code and $\mathcal{A}$ be a set of anticodes. For any $j \in \{1, \dots, \dim_{\mathbb{F}_q}(\mathcal{C})\}$, the ***j*-th generalized tensor weight** is

$$t_j(\mathcal{C}) := \min\left\{\dim_{\mathbb{F}_q}(A) : A \in \mathcal{A} \mid \dim_{\mathbb{F}_q}(\mathcal{C} \cap A) \geq j\right\}.$$

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If $\mathcal{A} = \mathcal{A}^{\mathrm{D}}$ then we recover the *generalized rank weights*. Indeed, we have

$$t_j^{\mathrm{D}}(\mathcal{C}) = \min\left\{\dim_{\mathbb{F}_q}(A) : A \in \mathcal{A}^{\mathrm{D}} \mid \dim_{\mathbb{F}_q}(\mathcal{C}\cap A) \geq j\right\} = m \cdot d_j(\mathcal{C})$$

for any $j \in \{1,\dots,\dim_{\mathbb{F}_q}(\mathcal{C})\}$.

Generalized weights: An anticode approach, *A. Ravagnani*
Journal of Pure and Applied Algebra, Elsevier, 2016.

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If $\mathcal{A} = \mathcal{A}^{\mathrm{ps}}$ then we recover the *generalized tensor ranks*. Indeed, we have

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for any $j \in \{1,\ldots,\dim_{\mathbb{F}_q}(\mathcal{C})\}$.

Tensor representation of rank-metric codes, *E. Byrne, A. Neri, A. Ravagnani, J. Sheekey*
SIAM Journal on Applied Algebra and Geometry, SIAM, 2019.

INVARIANTS FOR MATRIX CODES

Let $\mathcal{C}$ be a $[n \times m, k, d]_q$ code.

Proposition (Ravagnani - 2016)

The following hold.

(1) $t_1^{\mathrm{D}}(\mathcal{C}) = m\,d$,

(2) $t_k^{\mathrm{D}}(\mathcal{C}) \leq m\,n$,

(3) $t_j^{\mathrm{D}}(\mathcal{C}) \leq t_{j+1}^{\mathrm{D}}(\mathcal{C})$ for all $j \in \{1, \ldots, k-1\}$,

(4) $t_j^{\mathrm{D}}(\mathcal{C}) < t_{j+m}^{\mathrm{D}}(\mathcal{C})$ for all $j \in \{1, \ldots, k-m\}$,

(5) $t_j^{\mathrm{D}}(\mathcal{C}) \leq n - \left\lfloor \frac{k-j}{m} \right\rfloor$ for all $j \in \{1, \ldots, k\}$.

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We say that $\mathcal{C}$ is **MRD** if $m \mid k$ and $\mathcal{C}$ meets bound (5) for $j = 1$ with equality.

INVARIANTS FOR MATRIX CODES

Let $\mathcal{C}$ be a $[n \times m, k, d]_q$ code.

Proposition (Byrne, Neri, Ravagnani, Sheekey - 2019)

The following hold.

(1) $t_1^{\mathrm{ps}}(\mathcal{C}) = d$,

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(4) $t_j^{\mathrm{ps}}(\mathcal{C}) \geq d + j - 1$ for all $j \in \{1, \dots, k\}$,

(5) $t_j^{\mathrm{ps}}(\mathcal{C}) \leq \mathrm{trk}(\mathcal{C}) - k + j$ for all $j \in \{1, \dots, k\}$.

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Observe that $\mathcal{C}$ is **MTR** if $\mathcal{C}$ meets bound (4) for $j = k$ or (5) for $j = 1$ with equality.

INVARIANTS FOR MATRIX CODES

Let $\mathcal{C}$ be a $[n \times m, k, d]_q$ code.

Proposition (Byrne, C.)

The following hold.

(1) $t_1^{\text{cl}}(\mathcal{C}) = d^2$,

(2) $t_k^{\text{cl}}(\mathcal{C}) = \dim_{\mathbb{F}_q}(\text{colsupp}(\mathcal{C}))\dim_{\mathbb{F}_q}(\text{rowsupp}(\mathcal{C}))$,

(3) $t_j^{\text{cl}}(\mathcal{C}) \leq t_{j+1}^{\text{cl}}(\mathcal{C})$ for all $j \in \{1, \dots, k-1\}$,

(4) $t_j^{\text{ps}}(\mathcal{C}) \leq t_j^{\text{cl}}(\mathcal{C}) \leq t_j^{\text{D}}(\mathcal{C})$ or all $j \in \{1, \dots, k\}$.

AN EXAMPLE

Consider the following 1-dimensional Delsarte-Gabidulin codes over $\mathbb{F}_3$:

$$\mathcal{C} := \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \right\rangle_3,$$

$$\mathcal{D} := \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 2 & 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 2 & 2 & 1 & 1 \end{pmatrix} \right\rangle_3.$$

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One one check the following.

- $t_1^{\mathrm{ps}}(\mathcal{C}) = t_1^{\mathrm{ps}}(\mathcal{D}) = 2$.
- $t_j^{\mathrm{D}}(\mathcal{C}) = t_j^{\mathrm{D}}(\mathcal{D}) = 8$ for all $j \in \{1, \dots, 4\}$. In particular $\mathcal{C}$ and $\mathcal{D}$ are MRD.

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- $t_1^{\mathrm{cl}}(\mathcal{C}) = 4$, $\boldsymbol{t_2^{\mathrm{cl}}(\mathcal{C}) = 6}$ and $t_3^{\mathrm{cl}}(\mathcal{C}) = t_4^{\mathrm{cl}}(\mathcal{C}) = 8$.
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We can
distinguish more!

FURTHER QUESTIONS

- Find new classes of MTR codes.
- Determine the tensor rank of classes of matrix codes.
- Study properties of these invariants for *tensor codes*.

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"Matrices were created by God, tensors by Devil."

Max Noether

THANK YOU