Distributive laws between monads

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1 Distributive laws

A *distributive law* between two monads (S, η^S, μ^S) and (T, η^T, μ^T) on category $\mathcal{C}$ is a natural transformation

$$\lambda \quad : \quad S \circ T \quad \Rightarrow \quad T \circ S$$

such that the following diagrams commute

$$\begin{array}{ccccc} SST & \xRightarrow{S\lambda} & STS & \xRightarrow{\lambda S} & TSS \\ {\scriptstyle \mu^S T}\Downarrow & & & & \Downarrow{\scriptstyle T\mu^S} \\ ST & & \xRightarrow[\lambda]{} & & TS \end{array} \qquad \begin{array}{ccc} & T & \\ {\scriptstyle \eta^S T}\swarrow & & \searrow{\scriptstyle T\eta^S} \\ ST & \xRightarrow[\lambda]{} & TS \end{array}$$

$$\begin{array}{ccccc} STT & \xRightarrow{\lambda T} & TST & \xRightarrow{T\lambda} & TTS \\ {\scriptstyle S\mu^T}\Downarrow & & & & \Downarrow{\scriptstyle \mu^T S} \\ ST & & \xRightarrow[\lambda]{} & & TS \end{array} \qquad \begin{array}{ccc} & S & \\ {\scriptstyle S\eta^T}\swarrow & & \searrow{\scriptstyle \eta^T S} \\ ST & \xRightarrow[\lambda]{} & TS \end{array}$$

1. Can we always compose monads?
2. Draw those diagrams as string diagrams in the 2-category **Cat**.
3. Draw the laws for monads as string diagrams.
4. Show that the distributive law λ induces a structure of monad on the functor $T \circ S$.
5. Consider on **Set** the monads S of free monoid and T of free abelian group. Construct a distributive law $\lambda : ST \Rightarrow TS$ so that the composite monad is the monad of free ring.
6. How can we compose three (or more) monads with distributive laws?

2 Monads in Rel

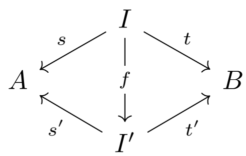
We define **Rel** as the 2-category whose 0-cells are sets, 1-cells $R : A \to B$ are relations $R \subseteq A \times B$, there is a unique 2-cell $\alpha : R \Rightarrow R' : A \to B$ whenever $R \subseteq R'$.

1. Recall both horizontal and vertical compositions in **Rel**.
2. Generalize the definition of adjunction and monad to any 2-category.
3. Show that a left adjoint in **Rel** is a function.
4. What is a monad in **Rel**?

3 Monads in Span

The 2-category of **Span** is the category where

- a 0-cell is a set
- a 1-cell from A to B is a *span*: $A \xleftarrow{s} I \xrightarrow{t} B$
- a 2-cell $f : (s,t) \to (s',t')$ is a function making the following diagram commute



Horizontal composition of 1-cells is given by pullback.

1. What is an endomorphism $A \to A$? A 2-cell between such endomorphisms?
2. Detail the compositions and identities of the 2-category.
3. Is it really a 2-category?
4. What is a monad in this "2-category"?
5. Generalize the definition of monad to any bicategory.

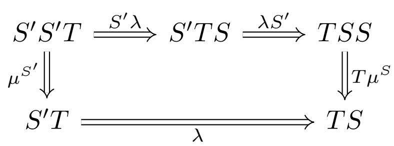
A *strict factorization system* on a category $\mathcal{C}$ consists of a pair of subcategories $\mathcal{L}$ and $\mathcal{R}$ with the same objects as $\mathcal{C}$ such that every morphism f of $\mathcal{C}$ factors uniquely as $f = r \circ l$ with $l \in \mathcal{L}$ and $r \in \mathcal{R}$.

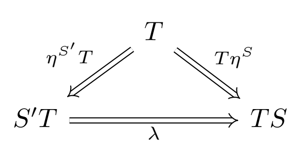
6. On **Set**, we write $\mathcal{L}$ (resp. $\mathcal{R}$) for the subcategory whose morphisms are epimorphisms (resp. monomorphisms). Show that these form a strict factorization system.
7. Show that a distributive law between monads in the bicategory **Span** corresponds to a strict factorization system.

4 Monads in monads

Given a 2-category $\mathcal{C}$, we write $\mathrm{Mnd}(\mathcal{C})$ for the 2-category whose

- 0-cells are monads in $\mathcal{C}$,
- a 1-cell from (C,S) to (C',S') is a *monad morphism*: a 1-cell $T : C \to C'$ of $\mathcal{C}$ together with a 2-cell $\lambda : S'T \Rightarrow TS$ such that

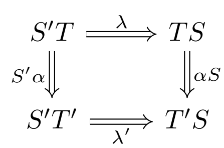




- a 2-cell

$$\alpha \quad : \quad (T,\lambda) \Rightarrow (T',\lambda') \quad : \quad (C,S) \to (C',S')$$

is a *monad transformation*: a 2-cell $\alpha : T \Rightarrow T'$ of $\mathcal{C}$ such that



8. Define the composition of 1-cells in $\mathrm{Mnd}(\mathcal{C})$.
9. Show that the distributive laws in a 2-category $\mathcal{C}$ correspond to monads in the 2-category of monads in $\mathcal{C}$.