

Algebras

Samuel Mimram

samuel.mimram@lix.polytechnique.fr

<http://lambdacat.mimram.fr>

November 29, 2021

1 Algebras for an endofunctor

An *algebra* for an endofunctor $F : \mathcal{C} \rightarrow \mathcal{C}$ is a pair (A, f) where A is an object of \mathcal{C} and $f : FA \rightarrow A$ a morphism of \mathcal{C} . A morphism $h : (A, f) \rightarrow (B, g)$ between two such algebras consists of a morphism $h : A \rightarrow B$ such that

$$\begin{array}{ccc} FA & \xrightarrow{Fh} & FB \\ f \downarrow & & \downarrow g \\ A & \xrightarrow{h} & B \end{array}$$

In the following, we mostly consider algebras in **Set**.

1. Define inductively the functions
 - `length` : 'a list -> int giving the length of a list,
 - `map` : ('a -> 'b) -> 'a list -> 'b list applying a function to all elements of a list,
 - `double` : 'a list -> 'a list which duplicates every successive element, for instance `double [1;2;3] = [1;1;2;2;3;3]`.
2. Suppose given a type 'a ilist of infinite lists with elements of type 'a. Define coinductively
 - `even` : 'a ilist -> 'a ilist keeping elements of a list at even positions,
 - `merge` : 'a ilist -> 'a ilist -> 'a ilist taking alternatively elements from one of two lists.
3. We write $S : \mathbb{N} \rightarrow \mathbb{N}$ for the successor function. Show that $[0, S] : 1 + \mathbb{N} \rightarrow \mathbb{N}$ is an initial algebra for the endofunctor $TX = 1 + X$ of **Set**.
4. Use this fact to define the function $h : \mathbb{N} \rightarrow \mathbb{Q}$ such that $h(n) = 2^{-n}$.
5. Show that two initial algebras of an endofunctor are isomorphic (via morphisms of algebras).
6. Show that an initial algebra $f : FA \rightarrow A$ of an endofunctor F is an isomorphism.
7. Solve the equation $x = 1 + ax$ and develop the solution in power series.
8. Show that the set $A^* = \bigoplus_{n \in \mathbb{N}} A^n$, which can be seen as the set of lists of elements of A , is an initial algebra for $TX = 1 + A \times X$.
9. Use this fact to define the length function $\ell : A^* \rightarrow \mathbb{N}$ and the double function $d : A^* \rightarrow A^*$. Show that $\ell \circ d(l) = 2\ell(l)$ for every $l \in A^*$.
10. Explain briefly how we could interpret simple inductive types of OCaml by using initial algebras.
11. What is the initial algebra for $TX = 1 + X \times X$? For $TX = X^*$? For $TX = A \times X$?

2 Coalgebras for an endofunctor

A *coalgebra* for $F : \mathcal{C} \rightarrow \mathcal{C}$ is a pair (A, f) with $f : A \rightarrow FA$. Morphisms are defined similarly as previously.

1. Show that the set $A^{\mathbb{N}}$ of *streams* is a final coalgebra for the endofunctor $TX = A \times X$.
2. Use this to define,
 - given $a \in A$, the constant stream equal to a ,
 - the function $\mathbb{N} \rightarrow A^{\mathbb{N}}$ which to n associates the stream $(n, n + 1, n + 2, \dots)$,
 - the function $A^{\mathbb{N}} \times A^{\mathbb{N}} \rightarrow A^{\mathbb{N}}$ which merges two streams,
 - the function $A^{\mathbb{N}} \rightarrow A^{\mathbb{N}}$ keeping even elements.
3. Show that final coalgebras are unique up to isomorphism and are isomorphisms.
4. Show that $\text{merge}(\text{even}(u), \text{odd}(u)) = u$ for every $u \in A^{\mathbb{N}}$, where $\text{odd}(l) = \text{even}(\text{tail}(l))$.

A *bisimulation* on $A^{\mathbb{N}}$ is a relation $R \subseteq A^{\mathbb{N}} \times A^{\mathbb{N}}$ such that $R(x :: u, x' :: u')$ implies $x = x'$ and $R(u, u')$. The *coinductive proof principle* says that if $R(u, u')$ for some bisimulation R then $u = u'$.

5. Assuming this principle, show again the result of previous question.
6. Show the coinductive proof principle (hint: show that R has a coalgebra structure).
7. Generalize the coinductive proof principle to an arbitrary endofunctor.
8. What is the final coalgebra of $TX = 1 + A \times X$? of $TX = 1 + X$?
9. Show that automatas can be seen as coalgebras.

References

- [1] Bart Jacobs and Jan Rutten. *An introduction to (co)algebra and (co)induction*, page 38–99. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2011.