Strong normalization of the simply-typed λ-calculus

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We recall the rules of the simply-typed λ-calculus:

$$\frac{}{\Gamma, x : A, \Gamma' \vdash x : A} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \Rightarrow B} \qquad \frac{\Gamma \vdash t : A \Rightarrow B \qquad \Gamma \vdash u : A}{\Gamma \vdash tu : B}$$

where, in the first rule, we suppose $x \notin \mathrm{dom}(\Gamma')$. We want to show that every typable term t (in an arbitrary context) is *strongly normalizable*, meaning that there is no infinite reduction from t.

1. Can we show the property by induction on the derivation of the typing of t?

In the course of the proof, will need the following *well-founded induction* principle.

2. Suppose given a set X equipped with a binary relation $\rightarrow$ which is *well-founded*: there is no infinite sequence of reductions. Suppose given a property P on the elements of X such that, for every $t \in X$, we have

$$\forall t \in X.\ ((\forall t' \in X.\ t \rightarrow t' \Rightarrow P(t')) \Rightarrow P(t))$$

   Show that $\forall t \in X.\ P(t)$ holds. How can we recover recurrence as a particular case of this?

A term t is *neutral* when no new redex is created when applied to another term u (all the redexes in tu are either in t or in u).

3. Give an explicit description of neutral terms.

We define $\mathcal{R}(A)$, the *reducible* terms of type A, by induction by

- $\mathcal{R}(A)$, for A atomic, is the set of strongly normalizable terms,
- $\mathcal{R}(A \Rightarrow B)$ is the set of terms t such that $tu \in \mathcal{R}(B)$ for every $u \in \mathcal{R}(A)$.

We are going to show that following conditions hold:

(CR1) if $t \in \mathcal{R}(A)$ then t is strongly normalizable,
(CR2) if $t \in \mathcal{R}(A)$ and $t \rightarrow t'$ then $t' \in \mathcal{R}(A)$,
(CR3) if t is neutral and for every t' such that $t \rightarrow t'$ we have $t' \in \mathcal{R}(A)$ then $t \in \mathcal{R}(A)$.

4. Show that these conditions imply that a variable x belongs to $\mathcal{R}(A)$ for every type A.
5. Show the conditions (CR1), (CR2) and (CR3) by induction on A.
6. Can we easily show that typable terms are reducible?
7. Suppose that $t[u/x] \in \mathcal{R}(B)$ for every $u \in \mathcal{R}(A)$. Show that $\lambda x.t \in \mathcal{R}(A \Rightarrow B)$.
8. Suppose that $x_1 : A_1, \ldots, x_n : A_n \vdash t : A$ is derivable. Show that for all $u_1 \in \mathcal{R}(A_1)$, ..., $u_n \in \mathcal{R}(A_n)$, we have $t[u_1/x_1, \ldots, u_n/x_n] \in \mathcal{R}(A)$.
9. Show that all typable terms are reducible.
10. Show that all typable terms are strongly normalizable.
11. Use this to show that typable terms are confluent.

References

[1] Jean-Yves Girard, Paul Taylor, and Yves Lafont. *Proofs and types*, volume 7. Cambridge university press Cambridge, 1989.