First-order quantifiers as adjoints

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1 Quantifiers as adjoints

Given a set X, we write $\mathcal{P}(X)$ for the associated powerset ordered by inclusion, which will be seen as a category. We can think of an element of $\mathcal{P}(X)$ as a predicate on X.

1. Explain how a function $f : X \to Y$ induces, by preimage, a functor $\Delta_f : \mathcal{P}(Y) \to \mathcal{P}(X)$.
2. Show that this functor admits a left adjoint $\exists_f : \mathcal{P}(X) \to \mathcal{P}(Y)$ and a right adjoint $\forall_f : \mathcal{P}(X) \to \mathcal{P}(Y)$.
3. Consider the function $f : \mathbb{N} \to \mathbb{B}$ where $\mathbb{B} = \{\text{even}, \text{odd}\}$ associating to a natural number its parity. What are the associated functions $\exists_f, \forall_f : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{B})$?
4. Explain how these functors can be used to model existential and universal quantification on a predicate $x : X, y : Y \vdash \phi(x, y)$.

2 Epis and monos

A morphism $m : A \to B$ is a *monomorphism* when for every pair of morphisms $f, g : X \to A$, we have $m \circ f = m \circ g$ implies $f = g$.

1. What is a monomorphism in the category **Set**?
2. Define the dual notion of *epimorphism*. What is it in **Set**?
3. Show that monomorphisms are closed under composition.

3 Subobjects

Given a category $\mathcal{C}$, an object A induces a category $\mathrm{Sub}(A)$ whose objects are pairs (U, m) with $U \in \mathcal{C}$ and $m : U \to A$ in $\mathcal{C}$, called *subobjects* of A, and whose morphisms $f : (U, m) \to (V, n)$ are morphisms $f : U \to V$ in $\mathcal{C}$ such that $n \circ h = m$.

1. Show that the category $\mathrm{Sub}(A)$ is a preorder.
2. Given a preorder $(X, \leq)$, one canonically associates a poset $(X/\sim, \leq)$ by quotienting X under the equivalence relation such that $x \sim y$ whenever $x \leq y$ and $y \leq x$. Given $A \in \mathbf{Set}$, what is the partial order obtained by quotienting $\mathbf{Set}(A)$?

4 Epi-mono factorization

A morphism $f : A \to B$ is *orthogonal* to a morphism $g : X \to Y$ in a category $\mathcal{C}$ when for every pair of morphisms $u : A \to X$ and $v : B \to Y$ such that $g \circ u = v \circ f$, there exists a unique morphism $h : B \to X$ such that $u = h \circ f$ and $v = g \circ h$:

$$\begin{array}{ccc} A & \xrightarrow{u} & X \\ {\scriptstyle f}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle g} \\ B & \xrightarrow[v]{} & Y \end{array}$$

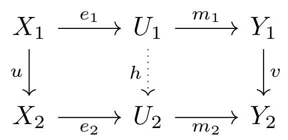
In this case, we write $f \perp g$. A *factorization system* $(\mathcal{E}, \mathcal{M})$ on a category $\mathcal{C}$ is a pair of collections of morphisms of $\mathcal{C}$ such that

- both $\mathcal{E}$ and $\mathcal{M}$ contain all isomorphisms and are closed under composition,
- every morphism f factors as $f = m \circ e$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$,
- every morphism $e \in \mathcal{E}$ is orthogonal to every morphism $m \in \mathcal{M}$.

1. Show that every function $f : X \to Y$ factors as $f = m \circ e$ for some surjective function $e : X \to U$ and injective function $m : U \to Y$.
2. Show that every surjective function $e : A \to B$ is orthogonal to every injective function $m : X \to Y$.
3. Construct a factorization system on **Set**.

We fix a category $\mathcal{C}$ with a factorization system $(\mathcal{E}, \mathcal{M})$.

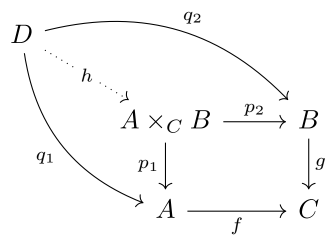
1. Show that for every commutative diagram as below, there exists a unique morphism $h : U_1 \to U_2$ making the diagram commute:



2. Given a morphism $f : A \to B$, construct a functor $\exists_f : \mathrm{Sub}(A) \to \mathrm{Sub}(B)$, and show that it defines the expected function in the case **Set**.

5 Pullbacks

In a category $\mathcal{C}$, a *pullback* of two morphisms $f : A \to C$ and $g : B \to C$ is an object $A \times_C B$ together with two morphisms $p_1 : A \times_C B \to A$ and $p_2 : A \times_C B \to B$ such that $f \circ p_1 = g \circ p_2$ and for every morphisms $q_1 : D \to A$ and $q_2 : D \to B$ such that $f \circ q_1 = g \circ q_2$ there exists a unique morphism $h : D \to A \times_C B$ making the following diagram commute



1. What is a pullback in **Set**?
2. Show that the pullback of a monomorphism along an arbitrary map is always a monomorphism.
3. Show that, in a category with pullbacks, every morphism $f : A \to B$ induces a functor $\Delta_f : \mathrm{Sub}(B) \to \mathrm{Sub}(A)$. What is such a function in **Set**?
4. Show that $\exists_f$ is left adjoint to Δ_f.