# Adjunctions: an alternative formulation

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October 11, 2021

## 1 An alternative formulation of adjunctions

1. Given a set $X$ and a monoid $M$, show that for every function $f : X \to GM$ there exists a unique morphism of monoids $f^* : X^* \to M$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & GM \\ {\scriptstyle\eta}\downarrow & \nearrow_{Gf^*} & \\ GX^* & & \end{array}$$

   where $G : \mathbf{Mon} \to \mathbf{Set}$ is the forgetful functor.

Our goal is to show that this situation generalizes to any adjunction. Suppose given a functor $G : \mathcal{D} \to \mathcal{C}$ between categories $\mathcal{C}$ and $\mathcal{D}$.

2. Show that, given objects $A$ in $\mathcal{C}$ and $F_A$ in $\mathcal{D}$, every map

$$\eta : A \to GF_A$$

   induces a natural transformation

$$\phi : \mathcal{D}(F_A, -) \Rightarrow \mathcal{C}(A, G-) : \mathcal{D} \to \mathbf{Set}$$

We say that a pair $(F_A, \eta)$ with $F_A \in \mathcal{D}$ and $\eta : A \to GF_A$ *represents* the functor $\mathcal{C}(A, G-) : \mathcal{D} \to \mathbf{Set}$ when the induced natural transformation $\phi : \mathcal{D}(F_A, -) \Rightarrow \mathcal{C}(A, G-)$ is a bijection, i.e. when $\phi_B : \mathcal{D}(F_A, B) \Rightarrow \mathcal{C}(A, GB)$ for every $B \in \mathcal{D}$.

3. Show that $(F_A, \eta)$ represents the functor $\mathcal{C}(A, G-) : \mathcal{D} \to \mathbf{Set}$ precisely when for every $B \in \mathcal{D}$ and morphism

$$f : A \to GB$$

   there exists a unique morphism

$$f^* : F_A \to B$$

   such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & GB \\ {\scriptstyle\eta}\downarrow & \nearrow_{Gf^*} & \\ GF_A & & \end{array}$$

4. Now, suppose that for every object $A$ of $\mathcal{C}$, there exists a pair $(F_A, \eta_A)$ representing the functor $\mathcal{C}(A, G-) : \mathcal{D} \to \mathbf{Set}$. Given $f : A \to A'$, construct a morphism $F_f : F_A \to F_{A'}$ in $\mathcal{D}$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & GF_A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle GF_f} \\ A' & \xrightarrow[\eta_{A'}]{} & GF_{A'} \end{array}$$

5. Construct a functor

$$\begin{aligned} F : \mathcal{C} &\to \mathcal{D} \\ A &\mapsto F_A \end{aligned}$$

6. Construct a natural bijection

$$\phi_{A,B} : \mathcal{D}(FA, B) \to \mathcal{C}(A, GB)$$

   What have we just shown?

7. Conversely, given an adjunction $F : \mathcal{C} \dashv \mathcal{D} : G$, show that for every object $A$ of $\mathcal{C}$ there is a pair $(FA, \eta_A)$ which represents the functor $\mathcal{C}(A, G-) : \mathcal{D} \to \mathbf{Set}$.

8. Apply the result to show that the forgetful functor $U : \mathbf{Mon} \to \mathbf{Set}$ admits a left adjoint.

# 2 Representable functors

A *coproduct* in a category $\mathcal{C}$ is a product in $\mathcal{C}^{\mathrm{op}}$.

1. Provide an explicit definition of coproducts.
2. What is a coproduct in **Set**, **Rel**, **Vect**?
3. Given an object $A$ of a category $\mathcal{C}$, describe the functor $\mathrm{Hom}(A, -) : \mathcal{C} \to \mathbf{Set}$.

A functor $F : \mathcal{C} \to \mathbf{Set}$ is *representable* when it is of the above form, i.e. , more precisely, there exists an object $A$ and a natural isomorphism $\phi : \mathrm{Hom}(A, -) \Rightarrow F$. We assume that this notion coincides with the one of previous section.

4. In a category $\mathcal{C}$, fix two objects $A$ and $B$ and consider the functor

$$\begin{aligned} F : \mathcal{C} &\to \mathbf{Set} \\ C &\mapsto \mathrm{Hom}(A, C) \times \mathrm{Hom}(B, C) \end{aligned}$$

   Complete the definition of the functor. Show that this functor is representable if and only if $A$ and $B$ admit a coproduct.

5. In the category **Vect**, given objects $A$ and $B$, what is a representation of the functor $\mathrm{Bilin}_{A,B} : \mathbf{Vect} \to \mathbf{Set}$ which to a vector space $C$ associates the set of bilinear functions $(A, B) \to C$?

6. Construct the diagonal functor

$$\begin{aligned} \Delta : \mathcal{C} &\to \mathcal{C} \times \mathcal{C} \\ A &\mapsto (A, A) \end{aligned}$$

   and show that it has a left adjoint precisely when $\mathcal{C}$ has coproducts.

7. When does a category have products?

8. Show that the notion of representability given in this part coincides with the one given in previous part. More precisely, given a functor $F : \mathcal{C} \to \mathbf{Set}$ and an object $A \in \mathcal{C}$, show that there is a bijection between
   - natural isomorphisms $\phi : \mathrm{Hom}(A, -) \Rightarrow F$,
   - elements $e \in FA$ such that for every $B \in \mathcal{C}$ and $f' \in FB$, there exists a unique $f : A \to B$ such that $(Ff)(e) = f'$.

   Explain how the second condition coincides with the definition of representability given in previous part.