Products, coproducts, pullbacks, monomorphisms

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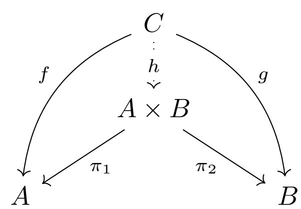
September 27, 2021

1 Cartesian products

Suppose fixed a category $\mathcal{C}$. We recall that a *cartesian product* of two objects A and B is given by an object $A \times B$ together with two morphisms

$$\pi_1 : A \times B \to A \qquad \text{and} \qquad \pi_2 : A \times B \to B$$

such that for every object C and morphisms $f : C \to A$ and $g : C \to B$, there exists a unique morphism $h : C \to A \times B$ making the diagram



commute. We also recall that a *terminal object* in a category is an object 1 such that for every object A there exists a unique morphism $f_A : A \to 1$. A category is *cartesian* when it has finite products, i.e. has a terminal object and every pair of objects admits a product.

1. Show that for every object A of a cartesian category, the objects $1 \times A$, A and $A \times 1$ are isomorphic.
2. How could you show previous questions using uniqueness of cartesian products proved in last session?
3. Show that the category **Rel** of sets and relations is cartesian.
4. Given a field $\Bbbk$, the category **Vect** of $\Bbbk$-vector spaces and linear functions is cartesian. Given a basis for A and B, what is a basis for $A \times B$?
5. Show that the category **Cat** is cartesian.
6. Given a cartesian category $\mathcal{C}$, show that the cartesian product induces a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

2 Coproducts

Notions in category theory can always be "dualized" in the following way.

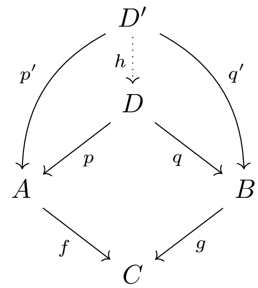
1. Given a category $\mathcal{C}$ define the category $\mathcal{C}^{\text{op}}$ obtained by reversing the morphisms.

A *co*something in a category $\mathcal{C}$ is a something in $\mathcal{C}^{\text{op}}$.

2. Show that **Set** is a cocartesian category, i.e. has coproducts and an initial object (an initial object is a coterminal object).
3. Show that the usual categories are cocartesian : **Set**, **Top**, **Rel**, **Vect**, **Cat**.

3 Pullbacks

Given two morphisms $f : A \to C$ and $g : B \to C$ with the same target, a *pullback* is given by an object D (sometimes abusively noted $A \times_C B$) together with two morphisms $p : D \to A$ and $q : D \to B$ such that $f \circ p = g \circ q$, and for every pair of morphisms $p' : D' \to A$ and $q' : D' \to B$ (with the same source) such that $f \circ p' = g \circ q'$, there exists a unique morphism $h : D' \to D$ such that $p \circ h = p'$ and $q \circ h = q'$.



1. What is a pullback in the case where C is the terminal object?
2. What is a pullback in **Set**?

A *pushout* in a category $\mathcal{C}$ is a pullback in $\mathcal{C}^{\mathrm{op}}$.

3. What is a pushout in **Set**? In **Top**?
4. Show that the pushout of an isomorphism is an isomorphism.

4 Monomorphisms

A *monomorphism* is a morphism $f : A \to B$ such that for every morphisms $g_1, g_2 : A' \to A$, we have that $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$:

$$A' \underset{g_2}{\overset{g_1}{\rightrightarrows}} A \xrightarrow{f} B$$

1. What is a monomorphism in **Set**?
2. Show that the pullback of a monomorphism along any morphism is a monomorphism.
3. Show that, in **Set**, the pushout of a monomorphism along any morphism is a monomorphism. Does this seem to be true in any category?
4. Define the dual notion of *epimorphism*. What is an epimorphism in **Set**?
5. In the category of posets, construct a morphism which is both a monomorphism and an epimorphism, but not an isomorphism.