Algebras for a monad

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1 Adjunctions for a monad

A *monad* (T, μ, η) is an endofunctor $T : \mathcal{C} \to \mathcal{C}$ equipped with two natural transformations $\mu : T \circ T \to T$ and $\eta : \mathrm{id}_\mathcal{C} \to T$ such that, for every $A \in \mathcal{C}$,

$$
\begin{array}{ccc}
TTTA & \xrightarrow{T\mu_A} & TTA \\
{\scriptstyle \mu_{TA}}\downarrow & & \downarrow{\scriptstyle \mu_A} \\
TTA & \xrightarrow[\mu_A]{} & TA
\end{array}
\qquad\text{and}\qquad
\begin{array}{ccccc}
TA & \xrightarrow{T\eta_A} & TTA & \xleftarrow{\eta_{TA}} & TA \\
 & {\scriptstyle \mathrm{id}_{TA}}\searrow & \downarrow{\scriptstyle \mu_A} & \swarrow{\scriptstyle \mathrm{id}_{TA}} & \\
 & & TA & &
\end{array}
$$

An *algebra* for a monad (T, μ, η) on a category $\mathcal{C}$ is a pair (A, a) with $a : TA \to A$ such that

$$
\begin{array}{ccc}
TTA & \xrightarrow{Ta} & TA \\
{\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle a} \\
TA & \xrightarrow[a]{} & A
\end{array}
\qquad\text{and}\qquad
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & TA \\
 & {\scriptstyle \mathrm{id}_A}\searrow & \downarrow{\scriptstyle a} \\
 & & A
\end{array}
$$

A morphism of T-algebras $f : (A, a) \to (B, b)$ is a morphism $f : A \to B$ in $\mathcal{C}$ such that

$$
\begin{array}{ccc}
TA & \xrightarrow{Tf} & TB \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\
A & \xrightarrow[f]{} & B
\end{array}
$$

Given a category $\mathcal{C}$ and T a monad on $\mathcal{C}$, we write $\mathcal{C}^T$ for the category of T-algebras.

1. Show that the forgetful functor $U : \mathbf{Mon} \to \mathbf{Set}$ has a left adjoint.
   What is the induced monad T on **Set**? What is its category of algebras?

   Solution. The left adjoint $F : \mathbf{Set} \to \mathbf{Mon}$ associates to a set A its free monoid A^*, whose elements are words $[a_1, \ldots, a_n]$ over A (we have already seen this). An algebra for the induced monad T on **Set** is a monoid. Namely, given an algebra (A, f), we can define a structure of monoid on A whose multiplication and unit are given by

   $$a \times b = f([a, b]) \qquad\qquad 1f([])$$

   Given three elements a, b, c, we have

   $$(a \times b) \times c = f[f([a, b]), c] = f[f([a, b]), f([c])] = f(\mu([[a, b], [c]])) = f([a, b, c])$$

   and similarly

   $$a \times (b \times c) = f([a, b, c])$$

   and the multiplication is associative. Moreover,

   $$1 \times a = f([f([]), a]) = f([f([]), f([a])]) = f(\mu([[], [a]])) = f([a]) = a$$

and similarly

$$a \times 1 = a$$

Conversely, the structure of algebra is entirely determined by these operations since one can show as above that

$$f([a_1, \ldots, a_n, a_{n+1}]) = f([f([a_1, \ldots, a_n]), a_{n+1}]) = f([a_1, \ldots, a_n]) \times a_{n+1}$$

and thus, by recurrence,

$$f([a_1, \ldots, a_n]) = (((1 \times a_1) \times a_2) \times \ldots) \times a_n$$

2. What are the algebras of the finite powerset monad on **Set**?

   Solution. Idempotent commutative monoids, i.e. unital semi-lattices.

3. Given a right adjoint functor $U : \mathcal{D} \to \mathcal{C}$, show that the category $\mathcal{D}$ is not always isomorphic to the category of algebras for the induced monad (hint: consider the forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$).

   Solution. Consider the forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$. The left adjoint $F : \mathbf{Set} \to \mathbf{Top}$ equips a set with the discrete topology. The induced monad $T : U \circ F$ is the identity monad on **Set**, whose algebras are sets (equipped with the identity morphism).

4. Given a monad $T : \mathcal{C} \to \mathcal{C}$, show that the forgetful functor $\mathcal{C}^T \to \mathcal{C}$ has a left adjoint, and that the induced monad is T.

   Solution. We define the left adjoint $F : \mathcal{C} \to \mathcal{C}^T$ as the functor which to a set A associates the *free algebra* (TA, μ_A), which is obviously an algebra by the axioms for monads. To a function $f : A \to B$, we associate the function $Tf : TA \to TB$, which is a morphism of algebras by naturality of μ. Given $A \in C$ and $B = (B, b) \in \mathcal{C}^T$, we define a bijection

$$\phi : \mathcal{C}^T(FA, B) \simeq \mathcal{C}(A, B) : \psi$$

as follows. Given a morphism of algebras $f : TA \to B$ in $\mathcal{C}^T$, we define

$$\phi(f) = \quad A \xrightarrow{\eta_A} TA \xrightarrow{f} B$$

and, given $f : A \to B$ in $\mathcal{C}$, we define

$$\psi(f) = \quad TA \xrightarrow{Tf} TB \xrightarrow{b} B$$

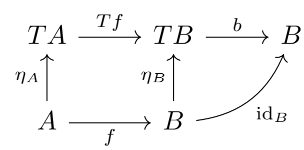
which is a morphism of algebras since

$$\begin{array}{ccccc} TTA & \xrightarrow{TTf} & TTB & \xrightarrow{Tb} & TB \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle \mu_B} & & \downarrow{\scriptstyle b} \\ TA & \xrightarrow[Tf]{} & TB & \xrightarrow[b]{} & B \end{array}$$

(left: naturality of μ, right: b is an algebra). Given $f : TA \to B$ in $\mathcal{C}^T$, we have $\psi \circ \phi(f) = f$ since

$$\begin{array}{ccccc} TA & \xrightarrow{T\eta_A} & TTA & \xrightarrow{Tf} & TB \\ & \searrow{\scriptstyle \mathrm{id}_{TA}} & \downarrow{\scriptstyle \mu_A} & & \downarrow{\scriptstyle b} \\ & & TA & \xrightarrow[f]{} & B \end{array}$$

(left: laws of monad, right: f is a morphism of algebras). Given $f : A \to B$ in $\mathcal{C}$, we have $\phi \circ \psi(f) = f$ since



The induced monad is the endofunctor $T = UF$. Its unit is

$$\phi(\mathrm{id}_{FA}) = \eta_A$$

Its multiplication is

$$U\psi(\mathrm{id}_{UFA}) = \mu_A$$

5. Construct the Kleisli category $\mathcal{C}_T$ associated to a monad. Show that the "forgetful functor" $\mathcal{C}_T \to \mathcal{C}$ has a left adjoint and that the induced monad is T.

   Solution. The forgetful functor $U : \mathcal{C}_T \to \mathcal{C}$ is defined by $UA = TA$ and, for $f : A \to TB$, $Uf = f \circ \eta_A$. The left adjoint $F : \mathcal{C} \to \mathcal{C}_T$ is defined by $FA = A$ and, for $f : A \to B$, $Ff = \eta_B \circ f$.

6. [Optional] Fix a monad T on $\mathcal{C}$ and consider the category whose objects are triples $(\mathcal{D}, F, G)$ with $F : \mathcal{C} \to \mathcal{D}$ left adjoint to $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F = T$, and whose morphisms $H : (\mathcal{D}, F, G) \to (\mathcal{D}', F', G')$ are functors $H : \mathcal{D} \to \mathcal{D}'$ such that $H \circ F = F'$ and $G' \circ H = G$. Show that the adjunctions associated to $\mathcal{C}^T$ and $\mathcal{C}_T$ are respectively terminal and initial in this category.

2 The Kleisli category as the category of free algebras

Our goal here is to show that the Kleisli category $\mathcal{C}_T$ associated to a monad T on $\mathcal{C}$ is the category of free algebras. A functor $F : \mathcal{C} \to \mathcal{D}$ is *full* (resp. *faithful*) when for every pair of objects A and B, the function

$$F_{A,B} : \mathcal{C}(A, B) \to \mathcal{D}(A, B)$$

is surjective (resp. injective).

7. Show that the "free algebra" functor $F : \mathcal{C}_T \to \mathcal{C}^T$ is full and faithful.

An *equivalence* between categories $\mathcal{C}$ and $\mathcal{D}$ consists in functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F \simeq \mathrm{id}_{\mathcal{C}}$ and $F \circ G \simeq \mathrm{id}_D$.

8. Show that an equivalence of categories is the same as a functor $F : \mathcal{C} \to \mathcal{D}$ which is essentially surjective (every object of $\mathcal{D}$ is isomorphic to one in the image of F) and full and faithful.

9. Show that the category $\mathcal{C}_T$ is equivalent to the full subcategory of $\mathcal{C}^T$ whose objects are free algebras.