Adjunctions

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We recall that a functor $U : \mathcal{D} \to \mathcal{C}$ has a left adjoint $F : \mathcal{C} \to \mathcal{D}$

$$\mathcal{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{D} \quad (F \dashv U)$$

when, for every objects $A \in \mathcal{C}$ and $B \in \mathcal{D}$ there is a bijection

$$\phi_{A,B} : \mathcal{D}(FA, B) \to \mathcal{C}(A, UB)$$

which is natural in A and B, in the sense that for every $f : A \to A'$ in $\mathcal{C}$ and $g : B \to B'$ in $\mathcal{D}$ we have the commutation of the diagram

$$\begin{array}{ccc}
\mathcal{D}(FA', B) & \xrightarrow{\phi_{A',B}} & \mathcal{C}(A', UB) \\
{\scriptstyle \mathcal{D}(Ff,g)}\downarrow & & \downarrow{\scriptstyle \mathcal{C}(f,Gg)} \\
\mathcal{D}(FA, B') & \xrightarrow[\phi_{A,B'}]{} & \mathcal{C}(A, UB')
\end{array}$$

1 Free monoids and categories

We write **Mon** for the category of monoids and morphisms of monoids.

1. Show that the forgetful functor U : **Mon** $\to$ **Set** admits a left adjoint F : **Set** $\to$ **Mon**.
2. Show that the forgetful functor U : **Cat** $\to$ **Graph** admits a left adjoint F : **Graph** $\to$ **Cat**.
3. Show that the forgetful functor U : **Top** $\to$ **Set** admits a left adjoint F : **Set** $\to$ **Top**.
4. Show that the forgetful functor U : **Top** $\to$ **Set** admits a right adjoint F : **Set** $\to$ **Top**.

2 The adjoint functor theorem between posets

In this exercise, we will study adjunctions between posets (seen as categories) and prove, in this restricted case, the adjoint functor theorem which provides conditions for the existence of adjoints to functors.

1. What is a functor between posets?
2. Consider the inclusion $F : \mathbb{Z} \to \mathbb{R}$ between posets. What is a left/right adjoint?
3. Consider the function $F : \mathbb{Z} \to \mathbb{Z}$ between posets such that $F(n) = 2n$. What is a left/right adjoint?

We suppose fixed a functor $F : \mathcal{C} \to \mathcal{D}$ where $\mathcal{C}$ and $\mathcal{D}$ are posets.

4. Show that if F is a left adjoint then it preserves arbitrary joins.
5. Suppose that $\mathcal{C}$ has all joins. Show that F is a left adjoint if and only if it preserves arbitrary joins.
6. A natural generalization of previous question is: a functor F is a left adjoint if and only if it preserves arbitrary colimits. Do you foresee any problem with proving this?

3 Terminal objects and products by adjunctions

1. Given a category $\mathcal{C}$, show that the terminal functor $T : \mathcal{C} \to \mathbf{1}$ has a right (resp. left) adjoint iff the category $\mathcal{C}$ admits a terminal (resp. initial) object.
2. Given a category $\mathcal{C}$, describe the *diagonal functor* $\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ and show that the category $\mathcal{C}$ admins cartesian products (resp. coproducts) iff the diagonal functor admits a right (resp. left) adjoint.

4 Quantifiers as adjoints

Given a set X, we write $\mathcal{P}(X)$ for the associated powerset ordered by inclusion, which will be seen as a category. We can think of an element of $\mathcal{P}(X)$ as a predicate on X.

1. Explain how a function $f : X \to Y$ induces, by preimage, a functor $\Delta_f : \mathcal{P}(Y) \to \mathcal{P}(X)$.
2. Show that this functor admits a left adjoint $\exists_f : \mathcal{P}(X) \to \mathcal{P}(Y)$ and a right adjoint $\forall_f : \mathcal{P}(X) \to \mathcal{P}(Y)$.
3. Consider the function $f : \mathbb{N} \to \mathbb{B}$ where $\mathbb{B} = \{even, odd\}$ associating to a natural number its parity. What are the associated functions $\exists_f, \forall_f : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{B})$?
4. Explain how these functors can be used to model existential and universal quantification on predicates.

5 Cartesian closed categories

A category is *cartesian closed* when for every object B, the functor $- \times B$ admits a right adjoint $B \Rightarrow -$.

1. Show that **Set** is cartesian closed.
2. Show that the category **POSet** of partially ordered sets and increasing functions is a cartesian closed category.
3. Show that the category **Mon** is cartesian but not closed (hint: look at the properties satisfied by the terminal object).