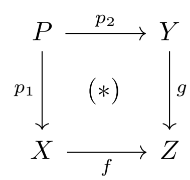
Travaux Dirigés
Pullbacks, monos, epis and subobjects

λ-calculs et catégories (23 septembre 2019)

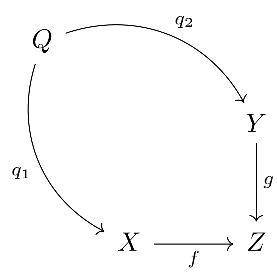
1 Pullbacks

In this exercise, we study the notion of pullback (called “produit fibré” in French), an important variation of the notion of “cartesian product” studied during the lecture. A commutative diagram in a category $\mathscr{C}$



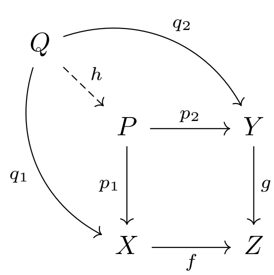
$$f \circ p_1 = g \circ p_2$$

is called a pullback diagram when the following property holds: for every commutative diagram



$$f \circ q_1 = g \circ q_2$$

there exists a unique morphism $h : Q \to P$ making the diagram below commute:



$$\begin{array}{rcl} f \circ p_1 &=& g \circ p_2 \\ f \circ q_1 &=& g \circ q_2 \\ q_1 &=& p_1 \circ h \\ q_2 &=& p_2 \circ h \end{array}$$

§1. Given two functions $f : X \to Z$ and $g : Y \to Z$, describe explicitly a set P and a pair of functions $p_1 : P \to X$ and $p_2 : P \to Y$ defining a pullback diagram of the form $(*)$ in the category Sets of sets and functions. Hint: the terminology “produit fibré” comes from this construction.

§2. Given two pullback diagrams

$$
\begin{array}{ccc}
Y'' & \xrightarrow{p'} & Y' \\
{\scriptstyle g''}\downarrow & (a) & \downarrow{\scriptstyle g'} \\
X'' & \xrightarrow[f']{} & X'
\end{array}
\qquad\qquad
\begin{array}{ccc}
Y' & \xrightarrow{p} & Y \\
{\scriptstyle g'}\downarrow & (b) & \downarrow{\scriptstyle g} \\
X' & \xrightarrow[f]{} & X
\end{array}
$$

in a category $\mathscr{C}$, show that the commutative diagram

$$
\begin{array}{ccccc}
Y'' & \xrightarrow{p'} & Y' & \xrightarrow{p} & Y \\
{\scriptstyle g''}\downarrow & & (c) & & \downarrow{\scriptstyle g} \\
X'' & \xrightarrow[f']{} & X' & \xrightarrow[f]{} & X
\end{array}
$$

obtained by "glueing" the two diagrams (a) and (b) defines a pullback diagram in the category $\mathscr{C}$.

§3. Suppose given three commutative diagrams $(a)(b)(c)$ in a category $\mathscr{C}$. We have seen in the previous question that when (b) is a pullback diagram,

$$(a) \text{ is a pullback diagram} \quad \Rightarrow \quad (c) \text{ is a pullback diagram}$$

Establish the converse property that

$$(c) \text{ is a pullback diagram} \quad \Rightarrow \quad (a) \text{ is a pullback diagram}$$

when (b) is a pullback diagram.

§4. Exhibit an example of three commutative diagrams $(a)(b)(c)$ such that

$$(a) \text{ and } (c) \text{ are pullback diagrams...} \quad \text{but } (b) \text{ is not a pullback diagram!}$$

Hint: one can take $X = \{x\}$ et $X'' = \{x''\}$ singleton sets and $X' = \{x_1, x_2\}$ a two-element set in the category $\mathscr{C} = \text{Sets}$.

2 Monomorphisms and epimorphisms

§1. An arrow $m : A \to B$ of a category $\mathscr{C}$ is called a monomorphism (mono for short) when m is left-simplifiable in the sense that

$$m \circ f = m \circ g \quad \Rightarrow \quad f = g$$

for every pair of arrows $f, g : X \to A$. Show that a function $m : A \to B$ is a mono in the category Sets precisely when it is an injective function.

§2. An arrow $e : A \to B$ of a category $\mathscr{C}$ is called an epimorphism (epi for short) when e is right-simplifiable in the sense that

$$f \circ e = g \circ e \quad \Rightarrow \quad f = g$$

for every pair of arrows $f, g : B \to Y$. Show that a function $e : A \to B$ is an epi in the category Sets precisely when it is a surjective function.

§3. Show that in any category $\mathscr{C}$, the composite $g \circ f : A \to C$ of two monos $f : A \to B$ and $g : B \to C$ is a mono, and that the composite of two epis is an epi.

§4. Show that an arrow $m : A \to B$ is a mono precisely when the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{id} & A \\ {\scriptstyle id}\big\downarrow & & \big\downarrow{\scriptstyle m} \\ A & \xrightarrow[m]{} & B \end{array}$$

is a pullback diagram in the category $\mathscr{C}$. Explain what the property means in the specific case of a function $m : A \to B$ in the category Sets.

§5. Show that every pullback diagram

$$\begin{array}{ccc} V & \xrightarrow{p} & U \\ {\scriptstyle m'}\big\downarrow & (\circledast) & \big\downarrow{\scriptstyle m} \\ B & \xrightarrow[f]{} & A \end{array}$$

in a category $\mathscr{C}$ satisfies the following property:

$$m : U \to A \text{ is a mono} \quad \Rightarrow \quad m' : V \to B \text{ is a mono.}$$

Show that the converse property does not hold by constructing a counter-example in the category Sets.

3 Comma categories and subobject categories

§1. Every object A in a category $\mathscr{C}$ induces a category $\mathscr{C}/A$ called the comma category on the object A, and defined in the following way. The objects of $\mathscr{C}/A$ are the pairs (X, f) consisting of an object $X \in \mathscr{C}$ and of an arrow

$$f : X \to A$$

with target A. The arrows of the category $\mathscr{C}/A$

$$h \quad : \quad (X, f) \quad \longrightarrow \quad (Y, g)$$

are the morphisms

$$h \quad : \quad X \quad \longrightarrow \quad Y$$

of the underlying category $\mathscr{C}$, making the diagram below commute:

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & A & \end{array}$$

Establish our claim above that $\mathscr{C}/A$ defines a category.

§2. Show that a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{p_2} & Y \\ {\scriptstyle p_1}\downarrow & (*) & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

in the category $\mathscr{C}$ is the same thing as a diagram

$$\begin{array}{ccccc} & & (P,u) & & \\ & \overset{p_1}{\swarrow} & (**) & \overset{p_2}{\searrow} & \\ (X,f) & & & & (Y,g) \end{array}$$

in the category $\mathscr{C}/Z$. Show moreover that the commutative diagram $(*)$ is a pullback in the category $\mathscr{C}$ precisely when the span diagram $(**)$ defines a cartesian product of (X, f) and (Y, g) in the comma category $\mathscr{C}/Z$. Deduce from this that the pullback diagram $(*)$ associated to a pair of morphisms $f : X \to Z$ and $g : Y \to Z$ is unique up to isomorphism.

§3. Every object A in a category $\mathscr{C}$ induces a category $\mathbf{Sub}(A)$ called the category of subobjects of A, and defined in the following way. Its objects (U, m) are the pairs consisting of an object $U \in \mathscr{C}$ and of a mono $m : U \to A$ with target A. Its morphisms $h : (U, m) \to (V, n)$ are the morphisms $h : U \to V$ of the underlying category $\mathscr{C}$ making the diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle m}\searrow & & \swarrow{\scriptstyle n} \\ & A & \end{array}$$

commute in the category $\mathscr{C}$. The category $\mathbf{Sub}(A)$ is thus the full subcategory of monos in the comma category $\mathscr{C}/A$. Show that the category $\mathbf{Sub}(A)$ is a preorder category, in the sense there exists at most one arrow $h : (U, m) \to (V, n)$ between two objects (U, m) and (V, n).

§4. Show that in the case $\mathscr{C} = \mathbf{Sets}$, one recovers the powerset $(\mathscr{P}(A), \subseteq)$ with subsets $U, V \subseteq A$ ordered by inclusion $U \subseteq V$, as the ordered set of equivalence classes associated to the preorder $\mathbf{Sub}(A)$. A useful convention in category theory is to identify the preorder category $\mathbf{Sub}(A)$ with the ordered set $(\mathscr{P}(A), \subseteq)$ in that case.

§5. A category $\mathscr{C}$ has pullbacks when there exists a pullback diagram $(*)$ for every pair of arrows $f : X \to Z$ and $g : Y \to Z$. Show that in a category $\mathscr{C}$ with pullbacks, every arrow $f : B \to A$ induces a monotone function

$$f^* \quad : \quad \mathbf{Sub}(A) \quad \longrightarrow \quad \mathbf{Sub}(B)$$

defined by transporting every mono $m : U \to A$ to the mono $m' : V \to B$ using the pullback diagram $(\circledast)$ in Exercise 2.5. Give an explicit description of the resulting monotone function

$$f^* \quad : \quad \mathscr{P}(A) \quad \longrightarrow \quad \mathscr{P}(B)$$

in the case when $\mathscr{C} = \mathbf{Sets}$ and when $f : A \to B$ is a function between two sets A and B.