λ-calculus: confluence, termination

We recall that λ-*terms* t are of the form x (a variable) or $\lambda x.t$ (an abstraction) or tu (an application). The β-*reduction* is the closure under context of the relation $(\lambda x.t)u \to t[u/x]$, i.e. the relation generated by

$$\frac{}{(\lambda x.t)u \to t[u/x]} \qquad \frac{t \to t'}{\lambda x.t \to \lambda x.t'} \qquad \frac{t \to t'}{tu \to t'u} \qquad \frac{u \to u'}{tu \to tu'}$$

We write $\xrightarrow{*}$ for the reflexive and transitive closure of $\to$.

1 Reduction graphs

The *reduction graph* of a λ-term t is the graph, whose vertices are λ-terms, defined as the smallest graph such that t is a vertex and there is an arrow between two vertices t and t' whenever $t \to t'$.

1. Write the respective reduction graphs of

$$(\lambda x.xx)(\lambda y.y)z \qquad \text{and} \qquad (\lambda xy.x)((\lambda x.xx)(\lambda xy.xy))$$

2. Can a reduction graph have loops?
3. Can a reduction graph be infinite?
4. Can a reduction graph be infinitely branching?

2 Confluence of the λ-calculus

Our goal is to show that the β-reduction is *confluent*, i.e. $u_1 \xleftarrow{*} t \xrightarrow{*} u_2$ implies that there exists v such that $u_1 \xrightarrow{*} v \xleftarrow{*} u_2$.

1. Show that β-reduction is *locally confluent*: $u_1 \leftarrow t \to u_2$ implies that there exists v such that $u_1 \xrightarrow{*} v \xleftarrow{*} u_2$.
2. Does local confluence imply confluence in general?

The *parallel reduction* $t \Rightarrow u$ on λ-terms is defined by:

$$\frac{}{x \Rightarrow x} \qquad \frac{t \Rightarrow t' \qquad u \Rightarrow u'}{(\lambda x.t)u \Rightarrow t'[u'/x]} \qquad \frac{t \Rightarrow t'}{\lambda x.t \Rightarrow \lambda x.t'} \qquad \frac{t \Rightarrow t' \qquad u \Rightarrow u'}{tu \Rightarrow t'u'}$$

3. Show that $\Rightarrow$ is reflexive.
4. Show that $\Rightarrow$ has the *diamond property*: $u_1 \Leftarrow t \Rightarrow u_2$ implies that there exists v such that $u_1 \Rightarrow v \Leftarrow u_2$.
5. Show that $\Rightarrow$ is confluent.
6. Show that $\to \subseteq \Rightarrow \subseteq \xrightarrow{*}$. Provide counter-examples showing that these inclusions are strict.
7. Conclude that $\to$ is confluent.

3 Termination of the simply typed λ-calculus

We recall the rules of the simply-typed λ-calculus:

$$\frac{}{\Gamma, x : A, \Gamma' \vdash x : A} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \Rightarrow B} \qquad \frac{\Gamma \vdash t : A \Rightarrow B \qquad \Gamma \vdash u : A}{\Gamma \vdash tu : B}$$

where, in the first rule, we suppose $x \notin \mathrm{dom}(\Gamma')$. We want to show that every typable term t (in an arbitrary context) is *strongly normalizable*, meaning that there is no infinite reduction from t.

1. Can we show the property by induction on the derivation of the typing of t?

In the course of the proof, will need the following *well-founded induction* principle.

2. Suppose given a set X equipped with a binary relation $\to$ which is *well-founded*: there is no infinite sequence of reductions. Suppose given a property P on the elements of X such that, for every $t \in X$, we have

$$\forall t \in X. \ ((\forall t' \in X. \ t \to t' \Rightarrow P(t')) \Rightarrow P(t))$$

   Show that $\forall t \in X. \ P(t)$ holds. How can we recover recurrence as a particular case of this?

We define $\mathcal{R}(A)$, the *reducible* terms of type A, by induction by

- $\mathcal{R}(A)$, for A atomic, is the set of strongly normalizable terms,
- $\mathcal{R}(A \Rightarrow B)$ is the set of terms t such that $tu \in \mathcal{R}(B)$ for every $u \in \mathcal{R}(A)$.

A term is *neutral* when it is not an abstraction. We are going to show that following conditions hold:

(CR1) if $t \in \mathcal{R}(A)$ then t is strongly normalizable,

(CR2) if $t \in \mathcal{R}(A)$ and $t \to t'$ then $t' \in \mathcal{R}(A)$,

(CR3) if t is neutral and for every t' such that $t \to t'$ we have $t' \in \mathcal{R}(A)$ then $t \in \mathcal{R}(A)$.

3. Show that these conditions imply that a variable x belongs to $\mathcal{R}(A)$ for every type A.
4. Show the conditions (CR1), (CR2) and (CR3) by induction on A.
5. Suppose that $t[u/x] \in \mathcal{R}(B)$ for every $u \in \mathcal{R}(A)$. Show that $\lambda x.t \in \mathcal{R}(A \Rightarrow B)$.
6. Suppose that $x_1 : A_1, \ldots, x_n : A_n \vdash t : A$ is derivable. Show that for all $u_1 \in \mathcal{R}(A_1)$, ..., $u_n \in \mathcal{R}(A_n)$, we have $t[u_1/x_1, \ldots, u_n/x_n] \in \mathcal{R}(A)$.
7. Show that all typable terms are reducible.
8. Show that all typable terms are strongly normalizable.
9. Use this to show that typable terms are confluent.