Distributive laws between monads

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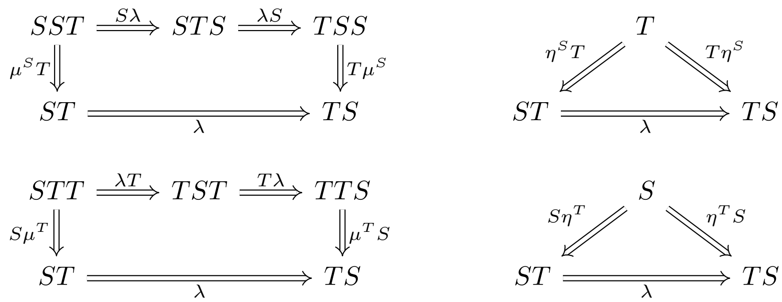
A *distributive law* between two monads (S, η^S, μ^S) and (T, η^T, μ^T) on category $\mathcal{C}$ is a natural transformation

$$\lambda \quad : \quad S \circ T \quad \Rightarrow \quad T \circ S$$

such that the following diagrams commute



1. Can we always compose monads?
2. Draw those diagrams as string diagrams in the 2-category **Cat**.
3. Draw the laws for monads as string diagrams.
4. Show that the distributive law λ induces a structure of monad on the functor $T \circ S$.
5. Consider on **Set** the monads S of free monoid and T of free abelian group. Construct a distributive law $\lambda : ST \Rightarrow TS$ so that the composite monad is the monad of free ring.
6. How can we compose three monads with distributive laws?

Suppose given two monads S and T as above. We write $U^S : \mathcal{C}^S \to \mathcal{C}$ for the forgetful functor from the category of algebras. A *lift* of the monad T to $\mathcal{C}^S$ is a monad $(\tilde{T}, \tilde{\eta}^T, \tilde{\mu}^T)$ such that

$$U^S\tilde{T} = TU^S \qquad\qquad U^S\tilde{\eta}^T = \eta^T U^S \qquad\qquad U^S\tilde{\mu}^T = \mu^T U^S$$

7. Show that distributive laws between S and T correspond to lifts of T to $\mathcal{C}^S$ (hint: for lift-to-distributive-law direction use the fact that U^S has a left adjoint such that the induced monad is S and notice that the first equality above can be seen as an invertible natural transformation).
8. [Optional] Show that the distributive laws in a 2-category $\mathcal{C}$ correspond to monads in the 2-category of monads in $\mathcal{C}$: $\mathrm{DLaw}(\mathcal{C}) = \mathrm{Mnd}(\mathrm{Mnd}(\mathcal{C}))$.

A *strict factorization system* on a category $\mathcal{C}$ consists of a pair of subcategories $\mathcal{L}$ and $\mathcal{R}$ with the same objects as $\mathcal{C}$ such that every morphism f of $\mathcal{C}$ factors uniquely as $f = r \circ l$ with $l \in \mathcal{L}$ and $r \in \mathcal{R}$.

9. Show that a distributive law between monads in the 2-category **Span** corresponds to a strict factorization system.