Coproducts, pullbacks, monoids

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1 Coproducts

Notions in category theory can always be "dualized" in the following way.

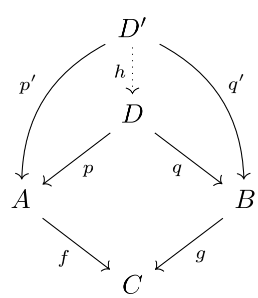
1. Given a category C define the category C^{op} obtained by reversing the morphisms.

A *co<u>something</u>* in a category C is a something in C^{op}.

2. Show that **Set** is a cocartesian category, i.e. has coproducts and an initial object (= a coterminal object).
3. Show that the usual categories are cocartesian : **Set**, **Rel**, **Top**, **Vect**, **Cat**.

2 Pullbacks

Given two morphisms $f : A \to C$ and $g : B \to C$ with the same target, a *pullback* is given by an object D (sometimes abusively noted $A \times_C B$) together with two morphisms $p : D \to A$ and $q : D \to B$ such that $f \circ p = g \circ q$, and for every pair of morphisms $p' : D' \to A$ and $q' : D' \to B$ (with the same source) such that $f \circ p' = g \circ q'$, there exists a unique morphism $h : D' \to D$ such that $p \circ h = p'$ and $q \circ h = q'$.



1. What is a pullback in the case where C is the terminal object?
2. What is a pullback in **Set**?

A *pushout* in a category $\mathcal{C}$ is a pullback in $\mathcal{C}^{\text{op}}$.

3. What is a pushout in **Set**? In **Top**?
4. Show that the pushout of an isomorphism is an isomorphism.

3 Monomorphisms

A *monomorphism* is a morphism $f : A \to B$ such that for every morphisms $g_1, g_2 : A' \to A$, we have that $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$:

$$A' \underset{g_2}{\overset{g_1}{\rightrightarrows}} A \xrightarrow{f} B$$

1. What is a monomorphism in **Set**?
2. Show that the pullback of a monomorphism along any morphism is a monomorphism.
3. Show that, in **Set**, the pushout of a monomorphism along any morphism is a monomorphism. Does this seem to be true in any category?
4. Define the dual notion of *epimorphism*. What is an epimorphism in **Set**?
5. In the category of posets, construct a morphism which is both a monomorphism and an epimorphism, but not an isomorphism.

4 (Co)monoids in cartesian categories

1. Given a cartesian category $\mathcal{C}$, show that the cartesian product induces a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$.
2. Generalize the definition of *monoid* to any cartesian category (a monoid in **Set** should be a monoid in the usual sense). When is a monoid commutative?
3. Generalize the notion of morphism of monoid.
4. A *comonoid* in $\mathcal{C}$ is a monoid in $\mathcal{C}^{\mathrm{op}}$. Make explicit the notion of comonoid.
5. What part of the cartesian structure on $\mathcal{C}$ did we really need in order to define the notion of monoid?
6. Show that in a cartesian category every object is a comonoid (with respect to product).
7. Given a category $\mathcal{C}$, show that the category of commutative comonoids and morphisms of comonoids in $\mathcal{C}$ is cartesian.