TD4 – Monads and algebras

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1 Algebras for a monad

An algebra for a monad (T, μ, η) on a category \mathcal{C} is a pair (A, a) with $a: TA \to A$ such that



A morphism of T-algebras $f: (A, a) \to (B, b)$ is a morphism $f: A \to B$ in C such that

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ a \downarrow & & \downarrow b \\ A & \xrightarrow{f} & B \end{array}$$

Given a category C and T a monad on C, we write C^T for the category of T-algebras.

1. Show that the forgetful functor $U : \mathbf{Mon} \to \mathbf{Set}$ has a left adjoint. What is the induced monad T on \mathbf{Set} ? What is its category of algebras?

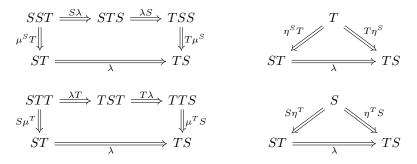
- 2. Given a right adjoint functor $U : \mathcal{D} \to \mathcal{C}$, the category \mathcal{D} is not always isomorphic to the category of algebras for T. Provide a counter-example (hint: consider the forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$).
- 3. Given a monad $T: \mathcal{C} \to \mathcal{C}$, show that the forgetful functor $\mathcal{C}^T \to \mathcal{C}$ has a left adjoint.
- 4. [Optional] Fix a monad T on C and consider the category whose objects are triples (\mathcal{D}, F, G) with $F : C \to \mathcal{D}$ left adjoint to $G : \mathcal{D} \to C$ such that $G \circ F = T$, and whose morphisms $H : (\mathcal{D}, F, G) \to (\mathcal{D}', F', G')$ are functors $H : \mathcal{D} \to \mathcal{D}'$ such that $H \circ F = F'$ and $G' \circ H = G$. Show that the above adjunction is a terminal object in this category.
- 5. [Optional] Show that the Kleisli category C_T is an initial object in this category.

2 Distributive laws between monads

A distributive law between two monads (S, η^S, μ^S) and (T, η^T, μ^T) on category C is a natural transformation

$$\lambda \hspace{0.1 in} : \hspace{0.1 in} S \circ T \hspace{0.1 in} \Rightarrow \hspace{0.1 in} T \circ S$$

such that the following diagrams commute



- 1. Draw those diagrams as string diagrams in the 2-category **Cat**.
- 2. Draw the laws for monads as string diagrams.
- 3. Show that the distributive law λ induces a structure of monad on the functor $T \circ S$.
- 4. Consider on **Set** the monads S of free monoid and T of free abelian group. Construct a distributive law $\lambda : ST \Rightarrow TS$ so that the composite monad is the monad of free ring.