TD4 – Monads and algebras

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1 Algebras for a monad

An *algebra* for a monad (T, μ, η) on a category $\mathcal{C}$ is a pair (A, a) with $a : TA \to A$ such that

$$\begin{array}{ccc} TTA & \xrightarrow{Ta} & TA \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle a} \\ TA & \xrightarrow[a]{} & A \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A & \xrightarrow{\eta_A} & TA \\ & {\scriptstyle \mathrm{id}_A}\searrow & \downarrow{\scriptstyle a} \\ & & A \end{array}$$

A morphism of T-algebras $f : (A, a) \to (B, b)$ is a morphism $f : A \to B$ in $\mathcal{C}$ such that

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A & \xrightarrow[f]{} & B \end{array}$$

Given a category $\mathcal{C}$ and T a monad on $\mathcal{C}$, we write $\mathcal{C}^T$ for the category of T-algebras.

1. Show that the forgetful functor $U : \mathbf{Mon} \to \mathbf{Set}$ has a left adjoint.
   What is the induced monad T on $\mathbf{Set}$?
   What is its category of algebras?
2. Given a right adjoint functor $U : \mathcal{D} \to \mathcal{C}$, the category $\mathcal{D}$ is not always isomorphic to the category of algebras for T. Provide a counter-example (hint: consider the forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$).
3. Given a monad $T : \mathcal{C} \to \mathcal{C}$, show that the forgetful functor $\mathcal{C}^T \to \mathcal{C}$ has a left adjoint.
4. [Optional] Fix a monad T on $\mathcal{C}$ and consider the category whose objects are triples $(\mathcal{D}, F, G)$ with $F : \mathcal{C} \to \mathcal{D}$ left adjoint to $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F = T$, and whose morphisms $H : (\mathcal{D}, F, G) \to (\mathcal{D}', F', G')$ are functors $H : \mathcal{D} \to \mathcal{D}'$ such that $H \circ F = F'$ and $G' \circ H = G$. Show that the above adjunction is a terminal object in this category.
5. [Optional] Show that the Kleisli category $\mathcal{C}_T$ is an initial object in this category.

2 Distributive laws between monads

A *distributive law* between two monads (S, η^S, μ^S) and (T, η^T, μ^T) on category $\mathcal{C}$ is a natural transformation

$$\lambda \quad : \quad S \circ T \quad \Rightarrow \quad T \circ S$$

such that the following diagrams commute

$$\begin{array}{ccccc} SST & \xRightarrow{S\lambda} & STS & \xRightarrow{\lambda S} & TSS \\ {\scriptstyle \mu^S T}\Downarrow & & & & \Downarrow{\scriptstyle T\mu^S} \\ ST & & \xRightarrow[\lambda]{\qquad\qquad} & & TS \end{array} \qquad \begin{array}{ccc} & T & \\ {\scriptstyle \eta^S T}\swarrow & & \searrow{\scriptstyle T\eta^S} \\ ST & \xRightarrow[\lambda]{\qquad} & TS \end{array}$$

$$\begin{array}{ccccc} STT & \xRightarrow{\lambda T} & TST & \xRightarrow{T\lambda} & TTS \\ {\scriptstyle S\mu^T}\Downarrow & & & & \Downarrow{\scriptstyle \mu^T S} \\ ST & & \xRightarrow[\lambda]{\qquad\qquad} & & TS \end{array} \qquad \begin{array}{ccc} & S & \\ {\scriptstyle S\eta^T}\swarrow & & \searrow{\scriptstyle \eta^T S} \\ ST & \xRightarrow[\lambda]{\qquad} & TS \end{array}$$

1. Draw those diagrams as string diagrams in the 2-category **Cat**.
2. Draw the laws for monads as string diagrams.
3. Show that the distributive law λ induces a structure of monad on the functor $T \circ S$.
4. Consider on **Set** the monads S of free monoid and T of free abelian group. Construct a distributive law $\lambda : ST \Rightarrow TS$ so that the composite monad is the monad of free ring.