TD3 – Graphs and the Yoneda lemma

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1 Graphs as presheaf categories

1. We write $\mathbf{G}$ for the category with two objects 0, 1 and four morphisms

$$\mathrm{id}_0 : 0 \to 0 \qquad s, t : 0 \to 1 \qquad \mathrm{id}_1 : 1 \to 1$$

   Show that the category $\mathbf{Cat}(\mathbf{G}^{\mathrm{op}}, \mathbf{Set})$ of functors and natural transformations defines the category of graphs (which are directed, which can contain multiple parallel edges and loops), which is usually denoted **Graph**.

2. Reformulate the definition of a category as a graph with some structure.

3. Explain that the category $\mathbf{Cat}(\mathbf{G}_2^{\mathrm{op}}, \mathbf{Set})$ of functors from the opposite of the category $\mathbf{G}_2$ with three objects 0, 1, 2 and nine morphisms

$$\mathrm{id}_0 : 0 \to 0 \qquad \mathrm{id}_1 : 1 \to 1 \qquad \mathrm{id}_2 : 2 \to 2 \qquad s_0, t_0 : 0 \to 1 \qquad s_1, t_1 : 1 \to 2 \qquad s, t : 0 \to 2$$

   with

$$s_0 \circ s_1 = s_0 \circ t_1 = s \qquad \text{and} \qquad t_0 \circ s_1 = t_0 \circ t_1 = t$$

   defines a category of 2-graphs and morphisms of 2-graphs.

4. Reformulate the definition of 2-categories using the notion of 2-graph.

Given a category $\mathcal{C}$, the category of *presheaves* $\hat{\mathcal{C}}$ is the category of functors $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ and natural transformations between them.

2 The Yoneda lemma

1. Define a graph Y_0 such that given a graph G, the vertices of G are in bijection with graph morphisms from Y_0 to G. Similarly, define a graph Y_1 such that we have a bijection between edges of G and graph morphisms from Y_1 to G.

2. Given a category $\mathcal{C}$, we define the *Yoneda functor* $Y : \mathcal{C} \to \hat{\mathcal{C}}$ by $YAB = \mathcal{C}(B, A)$ for objects $A, B \in \mathcal{C}$. Complete the definition of Y.

3. In the case of $\mathbf{G}$, what are the graphs obtained as the image of the two objects? A presheaf of the form YA for some object A is called a *representable* presheaf.

4. *Yoneda lemma:* show that for any category $\mathcal{C}$, presheaf $P \in \hat{\mathcal{C}}$, and object $A \in \mathcal{C}$, we have $P(A) \cong \hat{\mathcal{C}}(YA, P)$.

5. Show that the Yoneda embedding is full and faithful.

6. Show that the category of graphs (and more generally any presheaf category) is cartesian closed.