TD2 – Adjunctions

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1 Free monoids and categories

We write **Mon** for the category of monoids and morphisms of monoids.

1. Show that the forgetful functor $U : \mathbf{Mon} \to \mathbf{Set}$ admits a left adjoint $F : \mathbf{Set} \to \mathbf{Mon}$.
2. Show that the forgetful functor $U : \mathbf{Cat} \to \mathbf{Graph}$ admits a left adjoint $F : \mathbf{Graph} \to \mathbf{Cat}$.
3. Show that the forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ admits a left adjoint $F : \mathbf{Set} \to \mathbf{Top}$.
4. Show that the forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ admits a right adjoint $F : \mathbf{Set} \to \mathbf{Top}$.

2 Terminal objects and products by adjunctions

1. Given a category $\mathcal{C}$, show that the terminal functor $T : \mathcal{C} \to \mathbf{1}$ has a right (resp. left) adjoint iff the category $\mathcal{C}$ admits a terminal (resp. initial) object.
2. Given a category $\mathcal{C}$, describe the *diagonal functor* $\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ and show that the category $\mathcal{C}$ admins cartesian products (resp. coproducts) iff the diagonal functor admits a right (resp. left) adjoint.

3 Quantifiers as adjoints

Given a set X, we write $\mathcal{P}(X)$ for the associated powerset ordered by inclusion, which will be seen as a category. We can think of an element of $\mathcal{P}(X)$ as a predicate on X.

1. Explain how a function $f : X \to Y$ induces, by preimage, a functor $\Delta_f : \mathcal{P}(Y) \to \mathcal{P}(X)$.
2. Show that this functor admits a left adjoint $\exists_f : \mathcal{P}(X) \to \mathcal{P}(Y)$ and a right adjoint $\forall_f : \mathcal{P}(X) \to \mathcal{P}(Y)$.
3. Consider the function $f : \mathbb{N} \to \mathbb{B}$ where $\mathbb{B} = \{even, odd\}$ associating to a natural number its parity. What are the associated functions $\exists_f, \forall_f : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{B})$?
4. Explain how these functors can be used to model existential and universal quantification on predicates.

4 Cartesian closed categories

A category is *cartesian closed* when for every object B, the functor $- \times B$ admits a right adjoint $B \Rightarrow -$.

1. Show that **Set** is cartesian closed.

5 The exception monad

We write **pSet** for the category whose objects are *pointed sets*, i.e. pairs (A,a) where A is a set and $a \in A$, and morphisms $f : (A,a) \to (B,b)$ are functions such that $f(a) = b$. Here the distinguished element of the pointed set will be seen as a particular value indicating an error or an exception.

1. Describe the *forgetful functor* $U : \mathbf{pSet} \to \mathbf{Set}$ which to a pointed set associates the underlying set.

2. Construct a functor $F : \mathbf{Set} \to \mathbf{pSet}$ which is such that the sets $\mathbf{pSet}(FA, B)$ and $\mathbf{Set}(A, UB)$ are isomorphic.

3. Show that the families of isomorphisms

$$\varphi_{A,B} : \mathbf{pSet}(FA, B) \to \mathbf{Set}(A, UB) \qquad \text{and} \qquad \psi_{A,B} : \mathbf{Set}(A, UB) \to \mathbf{pSet}(FA, B)$$

described in previous question are natural. By "$\varphi_{A,B}$ is *natural*", we mean here that for every morphisms $f : A \to A'$ in **Set** and $h : B \to B'$ in **pSet** the diagram

$$\begin{array}{ccc} \mathbf{pSet}(FA', B) & \xrightarrow{\phi_{A',B}} & \mathbf{Set}(A', UB) \\ {\scriptstyle h\circ - \circ Ff}\downarrow & & \downarrow{\scriptstyle Uh \circ - \circ f} \\ \mathbf{pSet}(FA, B') & \xrightarrow[\phi_{A,B'}]{} & \mathbf{Set}(A, UB') \end{array}$$

commutes (in **Set**). Naturality of ψ is defined in a similar way.

4. We recall that a *monad* consists of an endofunctor $T : \mathcal{C} \to \mathcal{C}$ together with two natural transformations $\mu : T \circ T \Rightarrow T$ and $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow T$ such that the following diagrams commute:

$$\begin{array}{ccc} T \circ T \circ T & \xRightarrow{T\mu} & T \circ T \\ {\scriptstyle \mu_T}\Downarrow & & \Downarrow{\scriptstyle \mu} \\ T \circ T & \xRightarrow[\mu]{} & T \end{array} \qquad\qquad \begin{array}{ccccc} T & \xRightarrow{\eta_T} & T \circ T & \xLeftarrow{T\eta} & T \\ & {\scriptstyle \mathrm{id}_T}\searrow & \Downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \mathrm{id}_T} & \\ & & T & & \end{array}$$

Represent those diagrams using pasting diagrams in the 2-category **Cat**. Represent those diagrams using string diagrams.

5. Describe a structure of monad on $U \circ F$.

6. Given $f : A \to B$ an OCaml function which might raise an unique exception e and $g : B \to C$ a function which might raise an unique exception e', construct a function corresponding to the composite of f and g which might raise a unique exception e''.

7. We write $\mathbf{Set}_T$ the category whose objects are the objects of **Set** and morphisms $f : A \to B$ in $\mathbf{Set}_T$ are morphisms $f : A \to TB$ in **Set**. Compositions of two morphisms $f : A \to B$ and $g : B \to C$ in $\mathbf{Set}_T$ is defined by $g \circ f = \mu_C \circ Tg \circ f$ and identities are $\mathrm{id}_A = \eta_A$. Show that the axioms of categories are satisfied.

8. Give an explicit description of $\mathbf{Set}_T$.

9. A *non-deterministic function* is a function that might return a set of values instead of a single value. How could we could we similarly define a category of non-deterministic functions by a Kleisli construction?

10. Explain how the naturality condition of 3. is the usual naturality condition for φ seen as a natural transformation between the functors $\mathbf{pSet}(F-,-)$ and $\mathbf{Set}(-,U-)$ from $\mathbf{Set}^{\mathrm{op}} \times \mathbf{Set}$ to **Set**.