TD1 – Cartesian categories

Samuel Mimram
samuel.mimram@lix.polytechnique.fr

September 18, 2017

1 Categories and functors

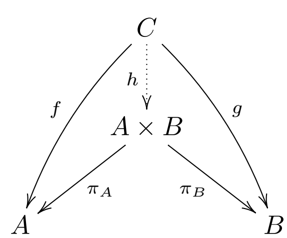
1. Recall the definition of *category* and provide some examples (e.g. **Set**, **Top**, **Vect**, **Grp**).
2. Recall the definition of a *functor* and provide some examples.
3. Define the category **Cat** of categories and functors.

2 Cartesian categories

Suppose fixed a category $\mathcal{C}$. A *cartesian product* of two objects A and B is given by an object $A \times B$ together with two morphisms

$$\pi_1 : A \times B \to A \qquad \text{and} \qquad \pi_2 : A \times B \to B$$

such that for every object C and morphisms $f : C \to A$ and $g : C \to B$, there exists a unique morphism $h : C \to A \times B$ making the diagram



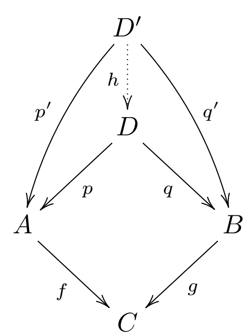
commute. We also recall that a *terminal object* in a category is an object 1 such that for every object A there exists a unique morphism $f_A : A \to 1$. A category is *cartesian* when it has finite products, i.e. has a terminal object and every pair of objects admits a product.

1. Suppose that $(E, \leq)$ is a poset. We associate to it category whose objects are elements of E and such that there exists a unique morphism between object a and b iff $a \leq b$. What is a terminal object and a product in this category?
2. Show that the category **Set** of sets and functions is cartesian.
3. Show that two terminal objects in a category are necessarily isomorphic.
4. Similarly, show that the cartesian product of two objects is defined up to isomorphism.
5. How could you show previous question using question 3.?
6. Show that for every object A of a cartesian category, the objects $1 \times A$, A and $A \times 1$ are isomorphic.
7. Show that for every objects A and B, $A \times B$ and $B \times A$ are isomorphic.

8. Show that for every objects A, B and C, $(A \times B) \times C$ and $A \times (B \times C)$ are isomorphic.
9. The notion of *coproduct* is dual to the notion of product, and the notion of *initial object* is dual to terminal object. Show that **Set** has all coproducts and an initial object (i.e. it is a co-cartesian category).
10. Show that the category **Rel** of sets and relations is cartesian.
11. We write **Vect** for the category of $\Bbbk$-vector spaces (where $\Bbbk$ is a fixed field) and linear functions. Show that this category is cartesian. Given a basis for A and B, describe a basis for $A \times B$.
12. Show that the category **Cat** is cartesian.

3 Pullbacks

Given two morphisms $f : A \to C$ and $g : B \to C$ with the same target, a *pullback* is given by an object D (sometimes abusively noted $A \times_C B$) together with two morphisms $p : D \to A$ and $q : D \to B$ such that $f \circ p = g \circ q$, and for every pair of morphisms $p' : D' \to A$ and $q' : D' \to B$ (with the same source) such that $f \circ p' = g \circ q'$, there exists a unique morphism $h : D' \to D$ such that $p \circ h = p'$ and $q \circ h = q'$.



1. What is a pullback in the case where C is the terminal object?
2. What is a pullback in **Set**?

4 Dual notions

A *coproduct* in a category $\mathcal{C}$ is a product in $\mathcal{C}^{\mathrm{op}}$.

1. What is a coproduct in **Set**? In **Rel**? In **Top**? In **Vect**?

A *pushout* in a category $\mathcal{C}$ is a pullback in $\mathcal{C}^{\mathrm{op}}$.

2. What is a pushout in **Set**? In **Top**?

5 (Co)monoids in cartesian categories

1. Generalize the definition of *monoid* to any cartesian category (a monoid in **Set** should be a monoid in the usual sense). When is a monoid commutative?
2. Generalize the notion of morphism of monoid.
3. A *comonoid* in $\mathcal{C}$ is a monoid in $\mathcal{C}^{\mathrm{op}}$. Make explicit the notion of comonoid.
4. Show that in a cartesian category every object is a comonoid.
5. Given a category $\mathcal{C}$, shown that the category of commutative comonoids and morphisms of comonoids in $\mathcal{C}$ is cartesian.