Travaux Dirigés

Distributivity laws between monads
Grothendieck construction and set-theoretic colimits

λ-calculs et catégories (12 décembre 2016)

1 Distributivity laws between monads

§1. Suppose given two categories $\mathscr{A}$ and $\mathscr{B}$, each of them equipped with a monad

$$(S, \mu_S, \eta_S) : \mathscr{A} \longrightarrow \mathscr{A} \qquad\qquad (T, \mu_T, \eta_T) : \mathscr{B} \longrightarrow \mathscr{B}$$

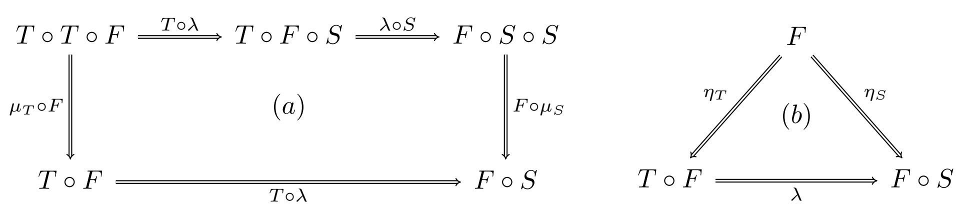
A homomorphism

$$(F, \lambda) \quad : \quad (\mathscr{A}, S) \longrightarrow (\mathscr{B}, T) \tag{1}$$

is defined as a functor $F : \mathscr{A} \to \mathscr{B}$ equipped with distributivity law

$$\lambda \quad : \quad T \circ F \Rightarrow F \circ S$$

making the diagrams of natural transformations below commute:



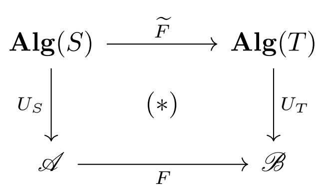
§1. Formulate the two commutative diagrams (a) and (b) as families of commutative diagrams between maps living in the category $\mathscr{B}$.

§2. Depict the commutative diagrams (a) and (b) in the language of string diagrams.

§3. Show that every homomorphism (F, λ) as in (1) induces a functor

$$\widetilde{F} \quad : \quad \mathbf{Alg}(S) \longrightarrow \mathbf{Alg}(T)$$

making the diagram below commute:



where U_S and U_T are the forgetful functors associated to the monads S and T, respectively.

§4. Conversely, show that every functor

$$\widetilde{F} \quad : \quad \mathbf{Alg}(S) \longrightarrow \mathbf{Alg}(T)$$

making the diagram $(*)$ commute induces a distributivity law $\lambda : T \circ F \Rightarrow F \circ S$ making the two diagrams (a) and (b) commute.

§5. Conclude that a homomorphism $(F, \lambda) : (\mathscr{A}, S) \to (\mathscr{B}, T)$ between two monads may be equivalently defined as a pair $(F, \widetilde{F})$ of functors

$$F : \mathscr{A} \longrightarrow \mathscr{B} \qquad\qquad \widetilde{F} : \mathbf{Alg}(S) \longrightarrow \mathbf{Alg}(T)$$

making the diagram $(*)$ commute.

§6. Deduce that there is a category **Mon** of monads and homomorphisms between them.

§7. Describe the free abelian group functor $F : \mathbf{Sets} \to \mathbf{Sets}$ which transports every set A to the free abelian group FA generated by the set A.

§8. Construct a family of functions

$$\lambda_A \quad : \quad TF(A) \longrightarrow FT(A)$$

parametrized by an object $A \in \mathscr{A}$ and check that the family λ is natural in A and makes the diagrams (a) and (b) commute.

§9. From this, deduce the existence of a functor

$$\widetilde{F} \quad : \quad \mathbf{Monoid} \longrightarrow \mathbf{Monoid}$$

from the category of monoids and homomorphisms, making the diagram below commute:

$$\begin{array}{ccc} \mathbf{Monoid} & \xrightarrow{\widetilde{F}} & \mathbf{Monoid} \\ {\scriptstyle U}\downarrow & (*) & \downarrow{\scriptstyle U} \\ \mathbf{Sets} & \xrightarrow[F]{} & \mathbf{Sets} \end{array}$$

§10. Describe the natural transformations μ_F and η_F equipping the functor F as a monad (F, μ_F, η_F).

§11. A distributivity law

$$\lambda \quad : \quad T \circ S \Rightarrow S \circ T$$

between two monads on the same category

$$(S, \mu_S, \eta_S) : \mathscr{A} \longrightarrow \mathscr{A} \qquad\qquad (T, \mu_T, \eta_T) : \mathscr{A} \longrightarrow \mathscr{A}$$

is a natural transformation making the diagrams below commute

$$\begin{array}{ccccc} T \circ T \circ S & \xrightarrow{T \circ \lambda} & T \circ S \circ T & \xrightarrow{\lambda \circ T} & S \circ T \circ T \\ {\scriptstyle \mu_T \circ S}\Downarrow & & (a) & & \Downarrow{\scriptstyle S \circ \mu_T} \\ T \circ S & & \xrightarrow[\lambda]{} & & S \circ T \end{array} \qquad \begin{array}{ccc} & S & \\ {\scriptstyle \eta_T}\swarrow & (b) & \searrow{\scriptstyle \eta_T} \\ T \circ S & \xrightarrow[\lambda]{} & S \circ T \end{array}$$

$$
\begin{array}{ccccc}
T\circ S\circ S & \xRightarrow{\lambda\circ S} & S\circ T\circ S & \xRightarrow{S\circ\lambda} & S\circ S\circ T \\
{\scriptstyle T\circ\mu_S}\Big\Downarrow & & (c) & & \Big\Downarrow{\scriptstyle \mu_S\circ T} \\
T\circ S & & \xRightarrow[\lambda]{\qquad\qquad} & & S\circ T
\end{array}
\qquad
\begin{array}{ccc}
 & T & \\
{\scriptstyle \eta_S}\swarrow & (d) & \searrow{\scriptstyle \eta_S} \\
T\circ S & \xRightarrow[\lambda]{\qquad} & S\circ T
\end{array}
$$

Depict the commutative diagrams (c) and (d) in the language of string diagrams.

§12. Show that every distributive law $\lambda : T\circ S \Rightarrow S\circ T$ between two monads S and T on the same category $\mathscr{A}$ induces a monad structure on the composite functor $S\circ T : \mathscr{A} \to \mathscr{A}$.

§13. Show that the natural transformation λ defined in §7. defines a distributivity law between the monads $S = F$ and T.

§14. Show that the monad $S\circ T$: **Sets** $\to$ **Sets** associated to the distributivity law $\lambda : T\circ S \Rightarrow S\circ T$ coincides with the free algebra monad (here, by algebra, we mean $\mathbb{Z}$-algebra).

2 Grothendieck construction and colimits computed in the category of sets and functions

We recall that a contravariant presheaf on a small category $\mathscr{C}$ is a functor

$$\varphi \quad : \quad \mathscr{C}^{\,op} \longrightarrow \mathbf{Sets}$$

Every contravariant presheaf φ induces a category $\mathbf{Groth}[\varphi]$ together with a projection functor

$$\pi[\varphi] \quad : \quad \mathbf{Groth}[\varphi] \longrightarrow \mathscr{C}. \tag{2}$$

The objects of the category are the pairs (c, x) with c an object of $\mathscr{C}$ and x an element of $\varphi(x)$; the maps

$$(c, x) \longrightarrow (d, y)$$

of the category are maps $f : c \to d$ of the underlying category $\mathscr{C}$ such that

$$\varphi(f)(y) = x.$$

§1. Show that these data define a category $\mathbf{Groth}[\varphi]$ together with a functor (2).

§2. Show that every natural transformation

$$\theta \quad : \quad \varphi \Rightarrow \phi \quad : \quad \mathscr{C}^{\,op} \longrightarrow \mathbf{Sets}$$

induces a functor

$$\mathbf{Groth}[\theta] \quad : \quad \mathbf{Groth}[\varphi] \longrightarrow \mathbf{Groth}[\psi]$$

making the diagram below commute

$$
\begin{array}{ccc}
\mathbf{Groth}[\varphi] & \xrightarrow{\mathbf{Groth}[\theta]} & \mathbf{Groth}[\psi] \\
{\scriptstyle \pi[\varphi]}\searrow & & \swarrow{\scriptstyle \pi[\psi]} \\
 & \mathscr{C} &
\end{array}
$$

§3. Conversely, show that every functor

$$F \quad : \quad \mathbf{Groth}[\varphi] \longrightarrow \mathbf{Groth}[\psi]$$

making the diagram below commute

$$\begin{array}{ccc} \mathbf{Groth}[\varphi] & \xrightarrow{\quad F \quad} & \mathbf{Groth}[\psi] \\ & {\scriptstyle \pi[\varphi]} \searrow \quad \swarrow {\scriptstyle \pi[\psi]} & \\ & \mathscr{C} & \end{array}$$

is of the form $F = \mathbf{Groth}[\theta]$ for a unique natural transformation

$$\theta \quad : \quad \varphi \Rightarrow \phi \quad : \quad \mathscr{C}^{op} \longrightarrow \mathbf{Sets}$$

§4. Construct a function

$$\theta_c \quad : \quad \varphi(c) \longrightarrow \pi_0(\mathbf{Groth}[\varphi])$$

for every object c of the category $\mathscr{C}$, where

$$\pi_0(\mathbf{Groth}[\varphi])$$

denotes the set of connected components of the category $\mathbf{Groth}[\varphi]$.

§5. Show that the diagram below commutes

$$\begin{array}{ccc} \varphi(c) & \xrightarrow{\quad \varphi(f) \quad} & \varphi(d) \\ & {\scriptstyle \theta_c} \searrow \quad \swarrow {\scriptstyle \theta_d} & \\ & \pi_0(\mathbf{Groth}[\varphi]) & \end{array}$$

for every map $f : c \to d$ in the category $\mathscr{C}$. Deduce from this that θ defines a natural transformation

$$\theta \quad : \quad \varphi \Rightarrow \pi_0(\mathbf{Groth}[\varphi])$$

and thus a cone.

§5. Show that the cone is a colimiting cone, and thus that colimit of the diagram

$$\varphi \quad : \quad \mathscr{C}^{op} \longrightarrow \mathbf{Sets}$$

coincides with the set

$$\pi_0(\mathbf{Groth}[\varphi])$$

of connected components of the Grothendieck category $\mathbf{Groth}[\varphi]$.