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Equalizers, epi-mono factorization, first-order logic

λ -calculs et catégories (17 octobre 2016)

1 Equalizers and coequalizers

In this exercise, we study the notion of equalizer (called “égalisateur” in French) and its dual notion of coequalizer. Suppose given a pair of cointial and cofinal arrows

$$f, g : X \rightrightarrows Y$$

in a category \mathcal{C} . An equalizer of f and g is an arrow $m : E \rightarrow X$ such that

$$f \circ m = g \circ m$$

and such that, for every arrow $n : F \rightarrow X$ such that

$$f \circ n = g \circ n$$

there exists a unique arrow $h : F \rightarrow E$ such that

$$n = m \circ h.$$

§1. Show that every pair of functions $f, g : X \rightarrow Y$ has an equalizer $m : E \rightarrow X$ in the category Sets and describe this equalizer.

§2. Show that when it exists in a category \mathcal{C} , the equalizer $m : E \rightarrow X$ of a pair of arrows $f, g : X \rightarrow Y$ is a mono.

§3. Formulate the dual notion of coequalizer $e : Y \rightarrow Q$ of two arrows

$$f, g : X \rightrightarrows Y$$

in a category \mathcal{C} .

§4. Show that when it exists in a category \mathcal{C} , the coequalizer $e : Y \rightarrow Q$ of two arrows $f, g : X \rightarrow Y$ is an epi.

§5. Show that every pair of functions $f, g : X \rightarrow Y$ have a coequalizer $e : Y \rightarrow Q$ in the category Sets and describe this coequalizer.

§6. One says that an epi $e : Y \rightarrow Q$ is regular when there exists a pair of arrows $f, g : X \rightarrow Y$ such that e is a coequalizer of f and g as in the diagram below:

$$X \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} Y \xrightarrow{e} Q$$

Show that every surjective function $e : A \rightarrow B$ is a regular epi in the category Sets.

2 Epi-mono factorization

An arrow $f : A \rightarrow B$ is orthogonal to an arrow $g : X \rightarrow Y$ in a category \mathcal{C} when for every pair of arrows $u : A \rightarrow X$ and $v : B \rightarrow Y$ making the diagram below commute

$$\begin{array}{ccc} A & \xrightarrow{u} & X \\ f \downarrow & & \downarrow g \\ B & \xrightarrow{v} & Y \end{array}$$

there exists a unique arrow $h : B \rightarrow X$ making the diagram below commute

$$\begin{array}{ccc} A & \xrightarrow{u} & X \\ f \downarrow & \nearrow h & \downarrow g \\ B & \xrightarrow{v} & Y \end{array}$$

in the sense that

$$u = h \circ f \quad \text{and} \quad v = g \circ h.$$

We write in that case

$$f \perp g.$$

A factorisation system $(\mathcal{E}, \mathcal{M})$ is a pair of collections \mathcal{E} and \mathcal{M} of arrows of the category \mathcal{C} satisfying the three properties below:

A. every arrow

$$X \xrightarrow{f} Y$$

of the category \mathcal{C} factors as

$$X \xrightarrow{e} U \xrightarrow{m} Y$$

where $e \in \mathcal{E}$ and $m \in \mathcal{M}$.

B. every arrow $e \in \mathcal{E}$ is orthogonal to every arrow $m \in \mathcal{M}$, what we write

$$\mathcal{E} \perp \mathcal{M}.$$

C. both collections \mathcal{E} and \mathcal{M} are closed under composition and contain the isos.

The purpose of the exercise is to show that the category **Sets** is equipped with a factorisation system $(\mathcal{E}, \mathcal{M})$ where \mathcal{E} and \mathcal{M} are respectively the collections of surjective and of injective functions.

§1. Show that every function

$$X \xrightarrow{f} Y$$

factors as

$$X \xrightarrow{e} U \xrightarrow{m} Y$$

where $e : X \rightarrow U$ is a surjective function and $m : U \rightarrow Y$ is an injective function.

§2. Show that every surjective function $e : A \rightarrow B$ is orthogonal to every injective function $m : X \rightarrow Y$ in the category **Sets**.

§3. Deduce from §1 and §2 that $(\mathcal{E}, \mathcal{M})$ defines a factorization system in **Sets**, where \mathcal{E} and \mathcal{M} are respectively the collections of surjective and injective functions in **Sets**.

§4. Suppose given a category \mathcal{C} equipped with a factorization system $(\mathcal{E}, \mathcal{M})$ and a commutative diagram

$$\begin{array}{ccccc} X_1 & \xrightarrow{e_1} & U_1 & \xrightarrow{m_1} & Y_1 \\ \downarrow u & & & & \downarrow v \\ X_2 & \xrightarrow{e_2} & U_2 & \xrightarrow{m_2} & Y_2 \end{array}$$

where $e_1, e_2 \in \mathcal{E}$ and $m_1, m_2 \in \mathcal{M}$. Show that there exists a unique arrow $h : U_1 \rightarrow U_2$ making the diagram below commute:

$$\begin{array}{ccccc} X_1 & \xrightarrow{e_1} & U_1 & \xrightarrow{m_1} & Y_1 \\ \downarrow u & & \downarrow h & & \downarrow v \\ X_2 & \xrightarrow{e_2} & U_2 & \xrightarrow{m_2} & Y_2 \end{array}$$

in the category \mathcal{C} .

§5. Suppose given a category \mathcal{C} whose collections \mathcal{E} of epis and \mathcal{M} of monos define a factorization system $(\mathcal{E}, \mathcal{M})$. Show that every arrow $f : X \rightarrow Y$ induces a subobject $(U, m) \in \text{Sub}(Y)$ defined as the unique subobject of Y such that the arrow $f : X \rightarrow Y$ factors as

$$X \xrightarrow{e} U \xrightarrow{m} Y$$

for a given epi $e : X \rightarrow U$. Show that in the case of the category **Sets**, the construction associates to every function $f : X \rightarrow Y$ its image in the set Y . For that reason, one often calls the subobject $m : U \rightarrow Y$ the image of the arrow $f : X \rightarrow Y$.

§6. Suppose that we are still in the situation of §5. Show that every arrow $f : A \rightarrow B$ of the category \mathcal{C} induces a monotone function

$$f_* : \text{Sub}(A) \longrightarrow \text{Sub}(B)$$

which transports every subobject (U, m) to the image $f_*(U, m)$ of the composite arrow

$$U \xrightarrow{m} A \xrightarrow{f} B$$

using the notion of “image” of an arrow $f \circ m : U \rightarrow B$ formulated in §5.

§7. Show that in the particular case $\mathcal{C} = \text{Sets}$, one associates in this way to every function $f : A \rightarrow B$ the monotone function

$$f_* : \mathcal{P}(A) \longrightarrow \mathcal{P}(B)$$

which transports every subset $U \subseteq A$ to its image $f(U) \subseteq B$.

§8. Suppose that we are in the situation of §5 and that the category \mathcal{C} has moreover pullbacks. We have seen in the previous TD that every arrow

$$f : A \longrightarrow B$$

induces in that case a monotone function

$$f^* : \mathbf{Sub}(B) \longrightarrow \mathbf{Sub}(A)$$

defined by “pulling back” subobjects $(V, n) \in \mathbf{Sub}(B)$ into subobjects $(U, m) \in \mathbf{Sub}(A)$. Show that the monotone function f_* is left adjoint to f^* in the sense that

$$f_*(U, m) \leq (V, n) \iff (U, m) \leq f^*(V, n)$$

for every pair of subobjects $(U, m) \in \mathbf{Sub}(A)$ and $(V, n) \in \mathbf{Sub}(B)$.

3 Application to first-order logic

Consider a family of sets X_1, \dots, X_n and their cartesian product $\Gamma = X_1 \times \dots \times X_n$. As we will see in the course, every first-order formula φ with free variables x_1, \dots, x_n induces a subset

$$[\varphi] \subseteq \Gamma$$

consisting of all the elements $(x_1, \dots, x_n) \in \Gamma$ satisfying the formula φ . Note that the interpretation $[\varphi]$ of the formula φ can be also seen as an element of the powerset:

$$[\varphi] \in \mathcal{P}(\Gamma).$$

§1. Every set X induces a function

$$\pi : \Gamma \times X \longrightarrow \Gamma$$

defined by the first projection. Given a first-order formula φ with free variables x_1, \dots, x_n , show that the subset

$$\pi^*[\varphi] = \{(x_1, \dots, x_n, x) \in \Gamma \times X \mid \varphi(x_1, \dots, x_n)\}$$

coincides with the interpretation of the same formula φ seen as a formula with free variables x_1, \dots, x_n, x .

§2. Given a first-order formula ψ with free variables x_1, \dots, x_n, x and with interpretation

$$[\psi] \in \mathcal{P}(\Gamma \times X)$$

show that

$$\pi_*[\psi] = \{(x_1, \dots, x_n) \in \Gamma \mid \exists x \in X, \psi(x_1, \dots, x_n, x)\}.$$

coincides with the interpretation $[\exists_{x \in X} \psi]$ of the formula $\exists_{x \in X} \psi$.

§3. From this, deduce that

$$[\exists_{x \in X} \psi] \leq_{\Gamma} [\varphi] \iff [\psi] \leq_{\Gamma \times X} [\varphi]$$

where we write $U \leq_{\Gamma} V$ for the inclusion $U \subseteq V$ between subsets $U, V \in \mathcal{P}(\Gamma)$. Justify this equivalence from the point of view of first-order logic.