TD2 – Graphs, adjunctions, monads

Samuel Mimram

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1 Graphs

1. Show that the category $\mathbf{Cat}(\mathbf{Gr}, \mathbf{Set})$ of functors and natural transformations from the category with two objects 0, 1 and four morphisms

$$\mathrm{id}_0 : 0 \to 0 \qquad \mathrm{id}_1 : 1 \to 1 \qquad s, t : 1 \to 0$$

   to the category $\mathbf{Set}$ of sets and functions defines the category of graphs, which is usually denoted $\mathbf{Graph}$.

2. Reformulate the definition of a category as a graph with some structure.

3. Explain that the category $\mathbf{Cat}(\mathbf{Gr}_2, \mathbf{Set})$ of functors from the category $\mathbf{Gr}_2$ with three objects 0, 1, 2 and nine morphisms

$$\mathrm{id}_0 : 0 \to 0 \qquad \mathrm{id}_1 : 1 \to 1 \qquad \mathrm{id}_2 : 2 \to 2 \qquad s_1, t_1 : 2 \to 1 \qquad s_0, t_0 : 1 \to 0 \qquad s, t : 2 \to 0$$

   with

$$s_0 \circ s_1 = s_0 \circ t_1 = s \qquad \text{et} \qquad t_0 \circ s_1 = t_0 \circ t_1 = t$$

   defines a category of 2-graphs and morphisms of 2-graphs.

4. Reformulate the definition of 2-categories using the notion of 2-graph.

2 Adjunctions between sets

We recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is *left adjoint* to a functor $G : \mathcal{D} \to \mathcal{C}$ if there is a natural bijection between $\mathcal{D}(FA, B)$ and $\mathcal{C}(A, GB)$.

1. Suppose given two functions $f : A \to B$ and $g : B \to A$ between sets A and B. Show that the two following properties are equivalent:

   (i) f and g are bijections and $f = g^{-1}$

   (ii) $\forall a \in A, \forall b \in B, \quad f(a) = b \quad \text{iff} \quad a = g(b)$

2. Conclude that an adjunction between two discrete categories is a bijection.

3 Free monoids and categories

We write $\mathbf{Mon}$ for the category of monoids and morphisms of monoids.

1. Show that the forgetful functor $U : \mathbf{Mon} \to \mathbf{Set}$ admits a left adjoint $F : \mathbf{Set} \to \mathbf{Mon}$.

2. Show that the forgetful functor $U : \mathbf{Cat} \to \mathbf{Graph}$ admits admits a left adjoint $F : \mathbf{Graph} \to \mathbf{Cat}$.

3. Show that the forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ admits admits a left adjoint $F : \mathbf{Set} \to \mathbf{Top}$.

4. Show that the forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ admits admits a right adjoint $F : \mathbf{Set} \to \mathbf{Top}$.

4 The exception monad

We write **pSet** for the category whose objects are *pointed sets*, i.e. pairs (A, a) where A is a set and $a \in A$, and morphisms $f : (A, a) \to (B, b)$ are functions such that $f(a) = b$. Here the distinguished element of the pointed set will be seen as a particular value indicating an error or an exception.

1. Describe the *forgetful functor* $U : \mathbf{pSet} \to \mathbf{Set}$ which to a pointed set associates the underlying set.

2. Construct a functor $F : \mathbf{Set} \to \mathbf{pSet}$ which is such that the sets $\mathbf{pSet}(FA, B)$ and $\mathbf{Set}(A, UB)$ are isomorphic.

3. Show that the families of isomorphisms

$$\varphi_{A,B} : \mathbf{pSet}(FA, B) \to \mathbf{Set}(A, UB) \qquad \text{and} \qquad \psi_{A,B} : \mathbf{Set}(A, UB) \to \mathbf{pSet}(FA, B)$$

   described in previous question are natural. By "$\varphi_{A,B}$ is *natural*", we mean here that for every morphisms $f : A \to A'$ in **Set** and $g : B \to B'$ in **pSet** the diagram

$$\begin{array}{ccc} \mathbf{pSet}(FA', B) & \xrightarrow{\phi_{A',B}} & \mathbf{Set}(A', UB) \\ {\scriptstyle g\circ - \circ Ff}\downarrow & & \downarrow{\scriptstyle Ug\circ - \circ f} \\ \mathbf{pSet}(FA, B') & \xrightarrow[\phi_{A,B'}]{} & \mathbf{Set}(A, UB') \end{array}$$

   commutes (in **Set**). Naturality of ψ is defined in a similar way.

4. We recall that a *monad* consists of an endofunctor $T : \mathcal{C} \to \mathcal{C}$ together with two natural transformations $\mu : T \circ T \Rightarrow T$ and $\eta : \mathrm{id}_\mathcal{C} \Rightarrow T$ such that the following diagrams commute:

$$\begin{array}{ccc} T\circ T\circ T & \xRightarrow{T\mu} & T\circ T \\ {\scriptstyle \mu_T}\Downarrow & & \Downarrow{\scriptstyle \mu} \\ T\circ T & \xRightarrow[\mu]{} & T \end{array} \qquad \begin{array}{ccccc} T & \xRightarrow{\eta_T} & T\circ T & \xLeftarrow{T\eta} & T \\ & {\scriptstyle \mathrm{id}_T}\searrow & \Downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \mathrm{id}_T} & \\ & & T & & \end{array}$$

   Represent those diagrams using pasting diagrams in the 2-category **Cat**. Represent those diagrams using string diagrams.

5. Describe a structure of monad on $U \circ F$.

6. Given $f : A \to B$ an OCaml function which might raise an unique exception e and $g : B \to C$ a function which might raise an unique exception e', construct a function corresponding to the composite of f and g which might raise a unique exception e''.

7. We write $\mathbf{Set}_T$ the category whose objects are the objects of **Set** and morphisms $f : A \to B$ in $\mathbf{Set}_T$ are morphisms $f : A \to TB$ in **Set**. Compositions of two morphisms $f : A \to B$ and $g : B \to C$ in $\mathbf{Set}_T$ is defined by $g \circ f = \mu_C \circ Tg \circ f$ and identities are $\mathrm{id}_A = \eta_A$. Show that the axioms of categories are satisfied.

8. Give an explicit description of $\mathbf{Set}_T$.

9. A *non-deterministic function* is a function that might return a set of values instead of a single value. How could we could similarly define a category of non-deterministic functions by a Kleisli construction?

10. Explain how the naturality condition of 3. is the usual naturality condition for φ seen as a natural transformation between the functors $\mathbf{pSet}(F-, -)$ and $\mathbf{Set}(-, U-)$ from $\mathbf{Set}^{\mathrm{op}} \times \mathbf{Set}$ to **Set**.

5 Terminal objects and products by adjunctions

1. Show that the category **Cat** has a terminal object **1**.

2. Given a category $\mathcal{C}$, show that the terminal functor $T : \mathcal{C} \to \mathbf{1}$ has a right (resp. left) adjoint iff the category $\mathcal{C}$ admits a terminal (resp. initial) object.

3. Given a category $\mathcal{C}$, describe the *diagonal functor* $D : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ and show that the category $\mathcal{C}$ admins cartesian products (resp. coproducts) iff the diagonal functor admits a right (resp. left) adjoint.

6 Monads generated by an adjunction

1. Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is left adjoint to a functor $G : \mathcal{D} \to \mathcal{C}$ iff there exists two natural transformations

$$\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F \quad \text{and} \quad \varepsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$$

respectively called the *unit* and *counit* of the adjunction, such that

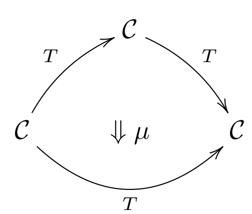
$$\varepsilon_F \cdot F\eta = \mathrm{id}_F \quad \text{and} \quad G\varepsilon \cdot \eta_G = \mathrm{id}_G \tag{1}$$

Describe the unit and counit corresponding the adjunctions studied in previous questions.

2. Show the above property.

3. Recall that a 2-category of categories, functors and natural transformations can be defined. What are the vertical and horizontal compositions in this category? What is the “exchange law” in a 2-category?

4. For every monad $T : \mathcal{C} \to \mathcal{C}$, the multiplication μ can be thus seen as a 2-cell



in this 2-category. By constructing the Poincaré dual of this diagram, we thus get a representation of the natural transformation μ using *string diagrams*. Similarly, give the string diagrammatic representation of the laws defining a monad as well as the laws (1).

5. Given an adjunction $(F, G, \eta, \varepsilon)$, show that the functor $G \circ F$ can be equipped with a structure of monad.

6. What are the monads associated the adjunction whose right adjoint is the forgetful functor from **pSet**/**Mon**/**Vect**/**Top** to **Set**?

7. Show that the forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ also admits a right adjoint.

8. [Optional] Show that if T is a monad on a category $\mathcal{C}$ then the category $\mathcal{C}$ is in adjunction with the category $\mathcal{C}_T$.

7 Monads in Rel

We define **Rel** as the 2-category whose 0-cells are sets, 1-cells $R : A \to B$ are relations $R \subseteq A \times B$, there is a unique 2-cell $\alpha : R \Rightarrow R' : A \to B$ whenever $R \subseteq R'$.

1. Recall both horizontal and vertical compositions in **Rel**.
2. Show that a left adjoint in **Rel** is a function.
3. What is a monad in **Rel**?

8 Monads in Haskell

Here is an excerpt of `http://www.haskell.org/haskellwiki/Monad`:

```
Monads can be viewed as a standard programming interface
to various data or control structures, which is captured
by the Monad class. All common monads are members of it:

class Monad m where
  (>>=) :: m a -> (a -> m b) -> m b
  return :: a -> m a

In addition to implementing the class functions, all
instances of Monad should obey the following equations:

return a >>= k  =  k a
m >>= return  =  m
m >>= (\x -> k x >>= h)  =  (m >>= k) >>= h
```

1. What does the `Maybe` monad defined below do?

```
data Maybe a = Nothing | Just a

instance Monad Maybe where
    return         = Just
    Nothing  >>= f = Nothing
    (Just x) >>= f = f x
```

2. What does the `List` monad defined below do?

```
instance Monad [] where
    m >>= f  = concatMap f m
    return x = [x]
```

3. A *Kleisli triple* $(T, \eta, (-)^*)$ on a category $\mathcal{C}$ consists of

   - a function $T : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C})$,
   - a function $\eta_A : A \to TA$ for every object A of $\mathcal{C}$,
   - a morphism $f^* : TA \to TB$ for every morphism $f : A \to TB$,

   such that for every objects A, B, C and morphisms $f : A \to TB$ and $g : B \to TC$,

$$\eta_A^* = \mathrm{id}_{TA} \qquad f^* \circ \eta_A = f \qquad g^* \circ f^* = (g^* \circ f)^*$$

   Show that Kleisli triples are in bijection with monads on $\mathcal{C}$.