TD1 – Cartesian categories

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21 September 2015

1 Categories and functors

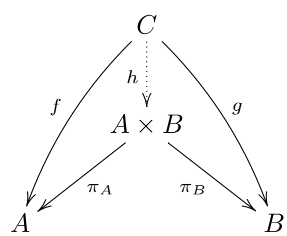
1. Recall the definition of *category* and provide some examples (e.g. **Set**, **Top**, **Vect**, **Grp**).
2. Recall the definition of a *functor* and provide some examples.
3. Define the category **Cat** of categories and functors.

2 Cartesian categories

Suppose fixed a category $\mathcal{C}$. A *cartesian product* of two objects A and B is given by an object $A \times B$ together with two morphisms

$$\pi_1 : A \times B \to A \qquad \text{and} \qquad \pi_2 : A \times B \to B$$

such that for every object C and morphisms $f : C \to A$ and $g : C \to B$, there exists a unique morphism $h : C \to A \times B$ making the diagram



commute. We also recall that a *terminal object* in a category is an object 1 such that for every object A there exists a unique morphism $f_A : A \to 1$. A category is *cartesian* when it has finite products, i.e. has a terminal object and every pair of objects admits a product.

1. Suppose that $(E, \leq)$ is a poset. We associate to it category whose objects are elements of E and such that there exists a unique morphism between object a and b iff $a \leq b$. What is a terminal object and a product in this category?
2. Show that the category **Set** of sets and functions is cartesian.
3. Show that two terminal objects in a category are necessarily isomorphic.
4. Similarly, show that the cartesian product of two objects is defined up to isomorphism.
5. How could you show previous question using question 3.?
6. Show that for every object A of a cartesian category, the objects $1 \times A$, A and $A \times 1$ are isomorphic.
7. Show that for every objects A and B, $A \times B$ and $B \times A$ are isomorphic.
8. Show that for every objects A, B and C, $(A \times B) \times C$ and $A \times (B \times C)$ are isomorphic.
9. The notion of *coproduct* is dual to the notion of product, and the notion of *initial object* is dual to terminal object. Show that **Set** has all coproducts and an initial object (i.e. it is a co-cartesian category).

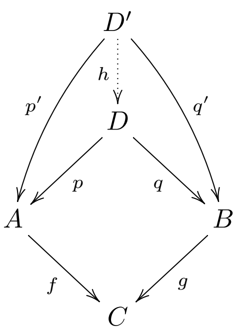
10. Show that the category **Rel** of sets and relations is cartesian.

11. We write **Vect** for the category of $\mathbb{k}$-vector spaces (where $\mathbb{k}$ is a fixed field) and linear functions. Show that this category is cartesian. Given a basis for A and B, describe a basis for $A \times B$.

12. Show that the category **Cat** is cartesian.

3 Pullbacks

Given two morphisms $f : A \to C$ and $g : B \to C$ with the same target, a *pullback* is given by an object D (sometimes abusively noted $A \times_C B$) together with two morphisms $p : D \to A$ and $q : D \to B$ such that $f \circ p = g \circ q$, and for every pair of morphisms $p' : D' \to A$ and $q' : D' \to B$ (with the same source) such that $f \circ p' = g \circ q'$, there exists a unique morphism $h : D' \to D$ such that $p \circ h = p'$ and $q \circ h = q'$.



1. What is a pullback in the case where C is the terminal object?

2. What is a pullback in **Set**?

4 Dual notions

A *coproduct* in a category $\mathcal{C}$ is a product in $\mathcal{C}^{\mathrm{op}}$.

1. What is a coproduct in **Set**? In **Rel**? In **Top**? In **Vect**?

A *pushout* in a category $\mathcal{C}$ is a pullback in $\mathcal{C}^{\mathrm{op}}$.

2. What is a pushout in **Set**? In **Top**?

5 (Co)monoids in cartesian categories

1. Generalize the definition of *monoid* to any cartesian category (a monoid in **Set** should be a monoid in the usual sense). When is a monoid commutative?

2. Generalize the notion of morphism of monoid.

3. A *comonoid* in $\mathcal{C}$ is a monoid in $\mathcal{C}^{\mathrm{op}}$. Make explicit the notion of comonoid.

4. Show that in a cartesian category every object is a comonoid.

5. Given a category $\mathcal{C}$, shown that the category of commutative comonoids and morphisms of comonoids in $\mathcal{C}$ is cartesian.

6 Representable graphs

1. Show that the category $\mathbf{Cat}(\mathbf{Gr}, \mathbf{Set})$ of functors and natural transformations from the category with two objects 0, 1 and four morphisms

$$\mathrm{id}_0 : 0 \to 0 \qquad \mathrm{id}_1 : 1 \to 1 \qquad s, t : 1 \to 0$$

to the category **Set** of sets and functions defines the category of graphs, which is usually denoted **Graph**.

2. Reformulate the definition of a category as a graph with some structure.

3. Explain that the category $\mathbf{Cat}(\mathbf{Gr}_2, \mathbf{Set})$ of functors from the category $\mathbf{Gr}_2$ with three objects 0, 1, 2 and nine morphisms

$$\mathrm{id}_0 : 0 \to 0 \qquad \mathrm{id}_1 : 1 \to 1 \qquad \mathrm{id}_2 : 2 \to 2 \qquad s_1, t_1 : 2 \to 1 \qquad s_0, t_0 : 1 \to 0 \qquad s, t : 2 \to 0$$

with

$$s_0 \circ s_1 = s_0 \circ t_1 = s \qquad \text{et} \qquad t_0 \circ s_1 = t_0 \circ t_1 = t$$

defines a category of 2-graphs and morphisms of 2-graphs.

4. Reformulate the definition of 2-categories using the notion of 2-graph.

Given a category $\mathcal{C}$, the category of *presheaves* $\hat{\mathcal{C}}$ is the category of functors $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ and natural transformations between them.

5. Define a graph Y_0 such that given a graph G, the vertices of G are in bijection with graph morphisms from Y_0 to G. Similarly, define a graph Y_1 such that we have a bijection between edges of G and graph morphisms from Y_1 to G.

6. Given a category $\mathcal{C}$, we define the *Yoneda functor* $Y : \mathcal{C} \to \hat{\mathcal{C}}$ by $YAB = \mathcal{C}(B, A)$ for objects $A, B \in \mathcal{C}$. Complete the definition of Y.

7. In the case of **Gr**, what are the graphs obtained as the image of the two objects? A presheaf of the form YA for some object A is called a *representable* presheaf.

8. *Yoneda lemma:* show that for any category $\mathcal{C}$, presheaf $P \in \hat{\mathcal{C}}$, and object $A \in \mathcal{C}$, we have $P(A) \cong \hat{\mathcal{C}}(YA, P)$.

9. Show that the Yoneda embedding is full and faithful.