TD7 – Presentations of monoidal categories

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1 Presentations of monoids and categories

Presentations of monoids are data which can provide small descriptions of monoids, from which many computations can be performed. The goal of this exercise is to introduce them and show how rewriting techniques can be useful in this context.

1. Suppose given a set G and a congruence $\equiv$ on G^* with respect to to concatenation. Show that the canonical structure of monoid on G^* induces a structure of monoid on the quotient set $G^*/{\equiv}$.

Given a monoid M, a pair (G, R) with $R \subseteq G^* \times G^*$ is called a *presentation* of M when there is an isomorphism of monoids $M \cong (G^*/{\equiv_R})$, where $\equiv_R$ is the smallest congruence containing R.

2. Propose (without proof) a presentation for the additive monoids $\mathbb{N}$, $\mathbb{N}/2\mathbb{N}$, $\mathbb{N} \times \mathbb{N}$ and $\mathbb{Z}$.

A *rewriting system* R on an alphabet G consists of a set of *rules* $R \subseteq G^* \times G^*$. We write $w \to w'$ when there exists u_1, v, v', u_2 such that $w = u_1 v u_2$, $w' = u_1 v' u_2$ and $(v, v') \in R$ and say that w *rewrites* to w'. We write $\overset{*}{\to}$ for the reflexive and transitive closure of $\to$.

3. Given a presentation (G, R), we can consider the elements of R as a string rewriting system on words over G. Show that when the rewriting system is convergent (i.e. both confluent and terminating), its normal forms are in bijection with elements of $G^*/{\equiv_R}$.

Recall that a *critical pair* is a word w of the form $w = v_1 u_1 = u_2 v_2$, such that v_1 is not a prefix of u_2, together with two rules (v_1, v_1') and (v_2, v_2') in R.

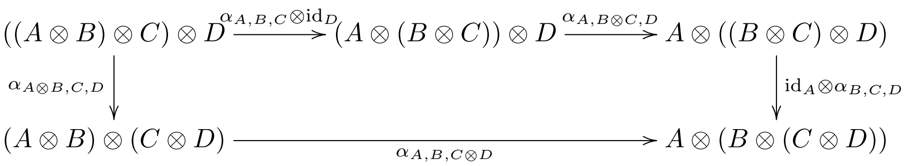
4. Show that for a terminating string rewriting system, confluence of critical pairs is equivalent to confluence of the whole system.
5. Show that the presentations proposed earlier are correct (notice that we require an isomorphism of monoids for a presentation!).
6. Show that every monoid admits at least one presentation.
7. What is a category with only one object? Generalize the notion of presentation from monoids to categories.
8. Propose a presentation for the category Δ with integers as objects and such that a morphism $f : m \to n$ is an increasing function $f : [m] \to [n]$ where $[n] = \{0, \ldots, n-1\}$.

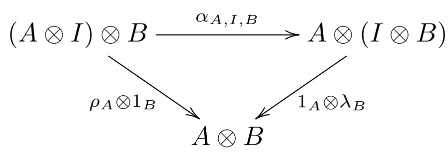
2 Monoids in monoidal categories

A *monoidal category* $(\mathcal{C}, \otimes, I)$ consists of a category $\mathcal{C}$ together with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called *tensor product*, an object $I \in \mathcal{C}$ called *unit* and three natural isomorphisms

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C) \qquad \lambda_A : I \otimes A \to A \qquad \rho_A : A \otimes I \to A$$

such that the following diagrams commute for every objects A, B, C and D:





A monoidal category is *strict* when α, λ and ρ are identity natural transformations.

1. Show that every cartesian category is a monoidal category.
2. Show that the operations $\otimes$ and $\oplus$ equip the category **Vect** with two different structures of monoidal category.
3. Generalize the notion of monoid to any monoidal category such that a monoid in $(\mathbf{Set}, \times, 1)$ is a monoid in the usual sense. What is a monoid in $(\mathbf{Vect}, \otimes, 1)$?
4. Provide an explicit definition of a strict monoidal category (without referring to α, λ and ρ).
5. Explain that a strict monoidal category is the same as a 2-category with only one 0-cell.
6. What is a monoid in $(\mathbf{Cat}, \times, 1)$?

3 A presentation of the simplicial category

1. Equip the category Δ with a structure of strict monoidal category.
2. Show that the object $[1]$ is terminal in Δ.
3. We write $\mu : [2] \to [1]$ and $\eta : [0] \to [1]$ for the terminal arrows. Show that the object $[1]$ is a monoid in Δ.
4. We write $\mathcal{M}$ for the free monoidal category containing a monoid object. How to formally express this as a universal property? Provide an explicit description of $\mathcal{M}$.
5. Propose a "presentation" for the monoidal category Δ.

In order to show that the previous presentation is correct, we are going to use the same methodology as before. We will show that every morphism in Δ can be represented as by a formal morphism in $\mathcal{M}$ and that equivalence classes of formal morphisms in $\mathcal{M}$ are in bijection with morphisms in Δ, by using canonical representatives provided by normal forms.

6. Define a surjective monoidal functor $F : \mathcal{M} \to \Delta_{\mathrm{inj}}$.
7. Orient the monoid axioms in order to obtain a locally confluent rewriting system (critical pairs can be computed in string diagrammatic notation).
8. We write $\mathbb{N}_* = \mathbb{N} \setminus \{0\}$. We define an order on functions $\mathbb{N}_*^p \to \mathbb{N}_*^q$ by $f < g$ iff $f(x) < g(x)$ (with respect to the product order) for every $x \in \mathbb{N}_*^p$. Show that this order is well-founded.
9. Show that the above rewriting system is terminating.
10. Conclude that the previous presentation is correct.

This methodology is in fact quite general and can be adapted to many other cases.

11. Propose a presentation of the category of natural numbers and bijections.
12. Propose a presentation of the category of natural numbers and functions.