TD5 – Realizability

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We write Λ for the set of λ-*terms* and Π for the set of *stacks* (sequences $t_1 \cdot t_2 \cdots t_n$ of λ-terms). *Processes* are elements (t, π) of $\Lambda \times \Pi$, often written $t \star \pi$.

1. Recall the definition of λ-terms, as well as reduction $\succ$ for the Krivine machine (operating on processes).

An element of $\mathcal{P}(\Pi)$ is called a *truth value*. Suppose fixed a set $\perp\!\!\!\perp$ closed under anti-reduction. A term $t \in \Lambda$ *realizes* a truth value $U \in \Pi$, what we write $t \Vdash U$ when $\forall \pi \in U, t \star \pi \in \perp\!\!\!\perp$.

2. We suppose fixed a set $\mathcal{T}$ of *generators* of given arity and write $\mathcal{T}^*$ for the set of generated terms. We also suppose fixed a set $\mathcal{R}$ of propositions of given first-order and second-order arities. The syntax of second order formulas is

$$A \quad ::= \quad X \quad | \quad R(A_1, \ldots, A_m, a_1, \ldots, a_n) \quad | \quad A \Rightarrow B \quad | \quad \forall x.A \quad | \quad \forall X.A$$

   where $a_i \in \mathcal{T}^*$ and $R \in \mathcal{R}$. Recall the rules of second order logic in natural deduction.

We define an interpretation $[\![A]\!] \in \mathcal{P}(\Pi)$ by induction on the formula A by

$$[\![A \Rightarrow B]\!] = \{t \cdot \pi \;/\; t \in |A|, \pi \in [\![B]\!]\} \qquad [\![\forall x.A]\!] = \bigcup_{a \in \mathcal{T}^*} [\![A[a/x]]\!] \qquad [\![\forall X.A]\!] = \bigcup_{V \in \mathcal{P}(\Pi)} [\![A[V/X]]\!]$$

where $|A| = \{t \in \Lambda \;/\; \forall \pi \in [\![A]\!], t \star \pi \in \perp\!\!\!\perp\}$ denotes the set of *realizers* of the formula A. Above, we have supposed fixed an interpretation of the first- and second-order free variables (by abuse of notation, given $V \in \mathcal{P}(\Pi)$, we still write V for a variable whose interpretation is V). We write $t \Vdash A$ when $t \in |A|$ and say that t *realizes* A.

3. Show that $t \Vdash A \Rightarrow B$ and $u \Vdash A$ implies $tu \Vdash B$.

4. Show that if for every $u \in \Lambda$, $u \Vdash A$ implies $tu \Vdash B$, then $\lambda x.tx \Vdash A \Rightarrow B$.

We admit the *adequation lemma*: if $x_1 : A_1, \ldots, x_n : A_n \vdash t : A$ is derivable and $\forall i, t_i \Vdash A_i$ then $t[t_1/x_1, \ldots, t_n/x_n] \Vdash A$.

5. Show that $\theta : \forall X.(X \Rightarrow X)$ implies that for every $(t, \pi) \in \Lambda \times \Pi$ we have $\theta \star t \cdot \pi \succ t \star \pi$ (use the closure by anti-reduction of $\{t \star \pi\}$ for $\perp\!\!\!\perp$).

6. We write $\mathrm{Bool}(x) = \forall X.X(0) \Rightarrow X(1) \Rightarrow X(x)$ (which is equivalent to $x = 0 \vee x = 1$). Show that $\vdash \theta : \mathrm{Bool}(0)$ implies $\theta \star t \cdot u \cdot \pi \succ t \star \pi$. And similarly for $\vdash \theta : \mathrm{Bool}(1)$.

7. We write $\exists x.A$ for the formula $\forall X.(\forall x.(A \Rightarrow X)) \Rightarrow X$. What is its interpretation? What is the interpretation of $\exists x.\,\mathrm{Bool}(x)$ if we suppose that $\mathcal{T}^* = \{0, 1\}$?

8. Define similarly the formulas $A \wedge B$ and $A \vee B$, $\perp$ and $\neg A$. What is their interpretation?

9. Given $U, V \in \mathcal{P}(\Pi)$, we write $U \leq V$ when there exists θ such that $\theta \Vdash U \Rightarrow V$. Show that $\leq$ is a preorder and $(\mathcal{P}(\Pi), \leq)$ a boolean algebra. Which one is it when $\perp\!\!\!\perp = \emptyset$?

10. Prove the adequation lemma.