

TD2 – Graphs, adjunctions, monads

Samuel Mimram

September 29, 2014

1 Graphs

1. Show that the category $\mathbf{Cat}(\mathbf{Gr}, \mathbf{Set})$ of functors and natural transformations from the category with two objects 0, 1 and four morphisms

$$\text{id}_0 : 0 \rightarrow 0 \quad \text{id}_1 : 1 \rightarrow 1 \quad s, t : 1 \rightarrow 0$$

to the category \mathbf{Set} of sets and functions defines the category of graphs, which is usually denoted \mathbf{Graph} .

2. Reformulate the definition of a category as a graph with some structure.
3. Explain that the category $\mathbf{Cat}(\mathbf{Gr}_2, \mathbf{Set})$ of functors from the category \mathbf{Gr}_2 with three objects 0, 1, 2 and nine morphisms

$$\text{id}_0 : 0 \rightarrow 0 \quad \text{id}_1 : 1 \rightarrow 1 \quad \text{id}_2 : 2 \rightarrow 2 \quad s_1, t_1 : 2 \rightarrow 1 \quad s_0, t_0 : 1 \rightarrow 0 \quad s, t : 2 \rightarrow 0$$

with

$$s_0 \circ s_1 = s_0 \circ t_1 = s \quad \text{et} \quad t_0 \circ s_1 = t_0 \circ t_1 = t$$

defines a category of 2-graphs and morphisms of 2-graphs.

4. Reformulate the definition of 2-categories using the notion of 2-graph.

2 Adjunctions between sets

We recall that a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is *left adjoint* to a functor $G : \mathcal{D} \rightarrow \mathcal{C}$ if there is a natural bijection between $\mathcal{D}(FA, B)$ and $\mathcal{C}(A, GB)$.

1. Suppose given two functions $f : A \rightarrow B$ and $g : B \rightarrow A$ between sets A and B . Show that the two following properties are equivalent:
 - (i) f and g are bijections and $f = g^{-1}$
 - (ii) $\forall a \in A, \forall b \in B, f(a) = b \text{ iff } a = g(b)$
2. Conclude that an adjunction between two discrete categories is a bijection.

3 Free monoids and categories

We write \mathbf{Mon} for the category of monoids and morphisms of monoids.

1. Define the forgetful functor $U : \mathbf{Mon} \rightarrow \mathbf{Set}$.
2. Show that this functor admits a left adjoint $F : \mathbf{Set} \rightarrow \mathbf{Mon}$.
3. Define the forgetful functor $U : \mathbf{Cat} \rightarrow \mathbf{Graph}$.
4. Show that this functor admits a left adjoint $F : \mathbf{Graph} \rightarrow \mathbf{Cat}$.

4 Terminal objects and products by adjunctions

1. Show that the category **Cat** has a terminal object **1**.
2. Given a category \mathcal{C} , show that the terminal functor $T : \mathcal{C} \rightarrow \mathbf{1}$ has a right (resp. left) adjoint iff the category \mathcal{C} admits a terminal (resp. initial) object.
3. Given a category \mathcal{C} , describe the *diagonal functor* $D : \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$ and show that the category \mathcal{C} admits cartesian products (resp. coproducts) iff the diagonal functor admits a right (resp. left) adjoint.

5 The exception monad

We write **pSet** for the category whose objects are *pointed sets*, i.e. pairs (A, a) where A is a set and $a \in A$, and morphisms $f : (A, a) \rightarrow (B, b)$ are functions such that $f(a) = b$. Here the distinguished element of the pointed set will be seen as a particular value indicating an error or an exception.

1. Describe the *forgetful functor* $U : \mathbf{pSet} \rightarrow \mathbf{Set}$ which to a pointed set associates the underlying set.
2. Construct a functor $F : \mathbf{Set} \rightarrow \mathbf{pSet}$ which is such that the sets $\mathbf{pSet}(FA, B)$ and $\mathbf{Set}(A, UB)$ are isomorphic.
3. Show that the families of isomorphisms

$$\varphi_{A,B} : \mathbf{pSet}(FA, B) \rightarrow \mathbf{Set}(A, UB) \quad \text{and} \quad \psi_{A,B} : \mathbf{Set}(A, UB) \rightarrow \mathbf{pSet}(FA, B)$$

described in previous question are natural. By “ $\varphi_{A,B}$ is *natural*”, we mean here that for every morphisms $f : A \rightarrow A'$ in **Set** and $g : B \rightarrow B'$ in **pSet** the diagram

$$\begin{array}{ccc} \mathbf{pSet}(FA', B) & \xrightarrow{\phi_{A',B}} & \mathbf{Set}(A', UB) \\ g \circ - \circ Ff \downarrow & & \downarrow Ug \circ - \circ f \\ \mathbf{pSet}(FA, B') & \xrightarrow{\phi_{A,B'}} & \mathbf{Set}(A, UB') \end{array}$$

commutes (in **Set**). Naturality of ψ is defined in a similar way.

4. We recall that a *monad* consists of an endofunctor $T : \mathcal{C} \rightarrow \mathcal{C}$ together with two natural transformations $\mu : T \circ T \Rightarrow T$ and $\eta : \text{id}_{\mathcal{C}} \Rightarrow T$ such that the following diagrams commute:

$$\begin{array}{ccc} T \circ T \circ T & \xrightarrow{T\mu} & T \circ T \\ \mu_T \downarrow & & \downarrow \mu \\ T \circ T & \xrightarrow{\mu} & T \end{array} \qquad \begin{array}{ccccc} T & \xrightarrow{\eta_T} & T \circ T & \xleftarrow{T\eta} & T \\ & \searrow \text{id}_T & \downarrow \mu & \swarrow \text{id}_T & \\ & & T & & \end{array}$$

Represent those diagrams using pasting diagrams in the 2-category **Cat**. Represent those diagrams using string diagrams.

5. Describe a structure of monad on $U \circ F$.
6. Given $f : A \rightarrow B$ an OCaml function which might raise a unique exception e and $g : B \rightarrow C$ a function which might raise a unique exception e' , construct a function corresponding to the composite of f and g which might raise a unique exception e'' .

7. We write \mathbf{Set}_T the category whose objects are the objects of \mathbf{Set} and morphisms $f : A \rightarrow B$ in \mathbf{Set}_T are morphisms $f : A \rightarrow TB$ in \mathbf{Set} . Compositions of two morphisms $f : A \rightarrow B$ and $g : B \rightarrow C$ in \mathbf{Set}_T is defined by $g \circ f = \mu_C \circ Tg \circ f$ and identities are $\text{id}_A = \eta_A$. Show that the axioms of categories are satisfied.
8. Give an explicit description of \mathbf{Set}_T .
9. A *non-deterministic function* is a function that might return a set of values instead of a single value. How could we define a category of non-deterministic functions by a Kleisli construction?
10. Explain how the naturality condition of 3. is the usual naturality condition for φ seen as a natural transformation between the functors $\mathbf{pSet}(F-, -)$ and $\mathbf{Set}(-, U-)$ from $\mathbf{Set}^{\text{op}} \times \mathbf{Set}$ to \mathbf{Set} .

6 Monads generated by an adjunction

1. Recall that a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is left adjoint to a functor $G : \mathcal{D} \rightarrow \mathcal{C}$ iff there exists two natural transformations

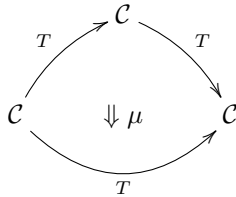
$$\eta : \text{id}_{\mathcal{C}} \rightarrow G \circ F \quad \text{and} \quad \varepsilon : F \circ G \rightarrow \text{id}_{\mathcal{D}}$$

respectively called the *unit* and *counit* of the adjunction, such that

$$\varepsilon_F \cdot F\eta = \text{id}_F \quad \text{and} \quad G\varepsilon \cdot \eta_G = \text{id}_G \tag{1}$$

Describe the unit and counit corresponding the adjunctions studied in previous questions.

2. Show the above property.
3. Recall that a 2-category of categories, functors and natural transformations can be defined. What are the vertical and horizontal compositions in this category? What is the “exchange law” in a 2-category?
4. For every monad $T : \mathcal{C} \rightarrow \mathcal{C}$, the multiplication μ can be thus seen as a 2-cell



in this 2-category. By constructing the Poincaré dual of this diagram, we thus get a representation of the natural transformation μ using *string diagrams*. Similarly, give the string diagrammatic representation of the laws defining a monad as well as the laws (1).

5. Given an adjunction $(F, G, \eta, \varepsilon)$, show that the functor $G \circ F$ can be equipped with a structure of monad.
6. What are the monads associated the adjunction whose right adjoint is the forgetful functor from $\mathbf{pSet}/\mathbf{Mon}/\mathbf{Vect}/\mathbf{Top}$ to \mathbf{Set} ?
7. Show that the forgetful functor $U : \mathbf{Top} \rightarrow \mathbf{Set}$ also admits a right adjoint.
8. [Optional] Show that if T is a monad on a category \mathcal{C} then the category \mathcal{C} is in adjunction with the category \mathcal{C}_T .

7 Monads in Rel

We define **Rel** as the 2-category whose 0-cells are sets, 1-cells $R : A \rightarrow B$ are relations $R \subseteq A \times B$, there is a unique 2-cell $\alpha : R \Rightarrow R' : A \rightarrow B$ whenever $R \subseteq R'$.

1. Recall both horizontal and vertical compositions in **Rel**.
2. Show that a left adjoint in **Rel** is a function.
3. What is a monad in **Rel**?

8 Monads in Haskell

Here is an excerpt of <http://www.haskell.org/haskellwiki/Monad>:

```
Monads can be viewed as a standard programming interface
to various data or control structures, which is captured
by the Monad class. All common monads are members of it:
```

```
class Monad m where
  (>>=) :: m a -> (a -> m b) -> m b
  return :: a -> m a
```

In addition to implementing the class functions, all instances of Monad should obey the following equations:

```
return a >>= k = k a
m >>= return = m
m >>= (\x -> k x >>= h) = (m >>= k) >>= h
```

1. What does the Maybe monad defined below do?

```
data Maybe a = Nothing | Just a
```

```
instance Monad Maybe where
  return = Just
  Nothing >>= f = Nothing
  (Just x) >>= f = f x
```

2. What does the List monad defined below do?

```
instance Monad [] where
  m >>= f = concatMap f m
  return x = [x]
```

3. A Kleisli triple $(T, \eta, (-)^*)$ on a category \mathcal{C} consists of

- a function $T : \text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{C})$,
- a function $\eta_A : A \rightarrow TA$ for every object A of \mathcal{C} ,
- a morphism $f^* : TA \rightarrow TB$ for every morphism $f : A \rightarrow TB$,

such that for every objects A, B, C and morphisms $f : A \rightarrow TB$ and $g : B \rightarrow TC$,

$$\eta_A^* = \text{id}_{TA} \quad f^* \circ \eta_A = f \quad g^* \circ f^* = (g^* \circ f)^*$$

Show that Kleisli triples are in bijection with monads on \mathcal{C} .