TD7 – Limits, presheaf categories

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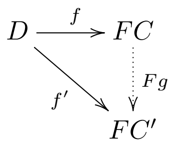
1 Limits

Suppose given a functor $F : \mathcal{C} \to \mathcal{D}$ and D and object of $\mathcal{D}$. An *universal arrow* from D to F is given by a pair (C, f) where C is an object of $\mathcal{C}$ and $f : D \to FC$ is a morphism in $\mathcal{D}$ such that for every other such pair (C', f') with $f' : D \to FC'$, there exists a unique morphism $g : C \to C'$ of $\mathcal{C}$ such that $Fg \circ f = f'$.



1. Suppose that $U : \mathcal{D} \to \mathcal{C}$ is a functor admitting a left adjoint $F : \mathcal{C} \to \mathcal{D}$. Show that for every object C of $\mathcal{C}$, (FC, η_C) is a universal arrow from C to U.

Suppose given two categories $\mathcal{J}$ and $\mathcal{C}$. The *diagonal functor* $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$ is such that

- given $C \in \mathcal{C}$, $\Delta(C)$ sends every object of $\mathcal{J}$ to C and every morphism of $\mathcal{J}$ to id_C,
- given $f : C \to D \in \mathcal{C}$, $\Delta(f)$ is the natural transformation whose components are f.

The *colimit* of a functor $F : \mathcal{J} \to \mathcal{C}$ is a universal arrow from F to Δ.

2. What is the colimit of a functor F in the case where $\mathcal{J}$ is the category with two objects and their respective identities?
3. What is the colimit of a functor F in the case where $\mathcal{J}$ is the empty category?
4. Express the notion of pushout as a colimit.
5. Show that any graph can be obtained as the colimit of a functor $F : \mathcal{J} \to$ **Graph** such that the image of an object is either G_0 (the graph with one vertex and no edge) or G_1 (the graph with two vertices and one edge between them).
6. Explain the dual notion of limit.
7. Show that a left adjoint preserves colimits.
8. Show that in a cartesian closed category with finite colimits, we have

$$A \Rightarrow (B \times C) \cong (A \Rightarrow B) \times (A \Rightarrow C) \qquad \text{and} \qquad A \times (B + C) \cong (A \times B) + (A \times C)$$

2 Simplicial sets

We write Δ for the category with $\mathbb{N}$ as objects and whose morphisms $f : m \to n$ are weakly increasing functions $f : [m] \to [n]$ where $[n] = \{0, 1, \ldots, n-1\}$.

1. Show that Δ is generated, as a category, by the morphisms

$$\eta_i^n : n \to n+1 \qquad \text{and} \qquad \mu_i^n : n+2 \to n+1$$

(with $0 \leq i \leq n$) defined by

$$\eta_i^n(k) = \begin{cases} k & \text{if } k < i \\ k+1 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \mu_i^n(k) = \begin{cases} k & \text{if } k \leq i \\ k-1 & \text{otherwise} \end{cases}$$

2. Find out the right members of the following equations satisfied by those morphisms:

$$\eta_j^{n+1} \circ \eta_i^n = ? \qquad\qquad \mu_j^{n+1} \circ \mu_i^{n+2} = ? \qquad\qquad \mu_j^{n+1} \circ \eta_i^{n+1} = ?$$

3. [Optional] Deduce a presentation of Δ as a category.

4. The *standard n-simplex* $\mathbf{\Delta}_n$ is the subspace of the euclidian space $\mathbb{R}^n$ whose points are

$$\mathbf{\Delta}_n \quad = \quad \left\{ (x_1, \ldots, x_n) \;\middle|\; x_i \geq 0 \quad \text{and} \quad \sum_i x_i = 1 \right\}$$

   We write Δ_+ for the full subcategory of Δ whose objects are strictly positive integers. Provide a geometrical interpretation of functors $\phi : \Delta_+^{\mathrm{op}} \to \mathbf{Set}$, which are called *simplicial sets*, by considering elements of $\phi(n)$ as standard n-simplices and morphisms $\phi(\varepsilon_i^n)$ as describing faces. What is the geometrical interpretation of $\phi(\eta_i^n)$? What is the geometrical interpretation of the above equations?

5. Give a description as simplicial sets of an empty square, a filled square, a (empty or filled) cube, a torus, a Möbius strip, etc.

3 Presheaf categories as free cocompletions

We recall that the category of presheaves $\hat{\mathcal{C}}$ over a category $\mathcal{C}$ is the category of functors $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ and natural transformations between them.

1. The Yoneda embedding $y : \mathcal{C} \to \hat{\mathcal{C}}$ is defined on objects $A \in \mathcal{C}$ by $yA = \mathcal{C}(-, A)$. Make completely explicit the definition of this functor.

2. [Yoneda lemma] Show that for every $A \in \mathcal{C}$ and $P \in \hat{\mathcal{C}}$, $\hat{\mathcal{C}}(yA, P) \cong P(A)$.
   Hint: show that $\phi \in \mathcal{C}(yA, P)$ is determined by $\phi_A(\mathrm{id}_A)$.

3. Deduce that the Yoneda embedding is full and faithful.

4. What is the Yoneda embedding of the objects in the case of graphs and simplicial sets?

5. What are products, coproducts, pushouts, pullbacks, etc. in the category of graphs?

6. Explain why every presheaf category is complete and cocomplete (assuming this for **Set**).

7. Describe a functor $I : \Delta \to \mathbf{Top}$ sending n to the canonical n-simplex.

8. Use this functor in order to build a *nerve* functor $N_I : \mathbf{Top} \to \hat{\Delta}$ associating to every topological space a simplicial set.

To any presheaf $P \in \hat{\mathcal{C}}$, we can associate a *category of elements* whose

- objects are pairs (A, a) with $A \in \mathcal{C}$ and $a \in P(A)$,
- and morphisms $f : (A, a) \to (B, b)$ are morphisms $f : A \to B$ of $\mathcal{C}$ such that $P(f)(b) = a$.

We write $\pi_P : \mathrm{El}(P) \to \mathcal{C}$ for the first projection functor. We define the *geometric realization* functor by

$$R_I(P) \quad = \quad \mathrm{colim}(\mathrm{El}(P) \xrightarrow{\pi_P} \Delta \xrightarrow{I} \mathbf{Top})$$

8. Compute the geometric realization of a simple simplicial set ($y3$ for instance).

9. Show that R_I is left adjoint to N_I.

10. Notice that the above proofs could be generalized to any functor $I : \mathcal{C} \to \mathcal{D}$ with $\mathcal{D}$ cocomplete and deduce that any presheaf $P \in \hat{\mathcal{C}}$ is canonically a colimit of representables:

$$P \quad = \quad \mathrm{colim}(\mathrm{El}(P) \xrightarrow{\pi_P} \mathcal{C} \xrightarrow{y} \hat{\mathcal{C}})$$

    We admit the following result: given a adjunction, the right adjoint is full and faithful if and only if the counit is an isomorphism.

11. Show that $\hat{\mathcal{C}}$ is the free cocompletion of $\mathcal{C}$: given a functor $F : \mathcal{C} \to \mathcal{D}$, there exists a unique cocontinuous functor $G : \hat{\mathcal{C}} \to \mathcal{D}$ such that $G \circ y = F$.