TD6 – Closed categories

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21 November 2013

1 Cartesian closed categories

Recall that a category $\mathcal{C}$ is *cartesian closed* when it has finite products and for every object B of $\mathcal{C}$ the functor $- \times B : \mathcal{C} \to \mathcal{C}$ admits a right adjoint, written $(-)^B$, i.e. there exists a bijection $\mathcal{C}(A \times B, C) \cong \mathcal{C}(A, C^B)$ natural in A and C.

1. Show that the category **Set** is cartesian closed.
2. Describe the unit and the counit of the adjunction defining the closure. Check that the laws between the unit and counit in an adjunction are satisfied.
3. Show that the category **Cat** is cartesian closed.

2 Simply typed λ-calculus

Recall the syntax of λ-terms with products:

$$M \quad ::= \quad x \quad | \quad \lambda x.M \quad | \quad MM \quad | \quad (M, M) \quad | \quad \pi_1 \quad | \quad \pi_2 \quad | \quad ()$$

and the syntax of types:

$$A \quad ::= \quad a \quad | \quad A \times A \quad | \quad 1$$

The λ-terms will be considered modulo α-conversion (renaming of bound variables). The β-conversion rules are defined by

$$(\lambda x.M)N \equiv_\beta M[N/x] \qquad\qquad \pi_i(M_1, M_2) \equiv_\beta M_i$$

and those of η-conversion by

$$\lambda x.Mx \equiv_\eta M \qquad\qquad (\pi_1 M, \pi_2 M) \equiv_\eta M \qquad\qquad M \equiv_\eta () \text{ if } M \text{ has type } 1$$

1. Recall the typing rules.
2. Define the substitution operation $M[N/x]$ by induction on the structure of the term M. Show that typing is preserved under β-reduction.
3. We want to make explicit context manipulations. Which rules do we have to add if we want to obtain an equivalent deduction system where the axiom and unit rule have been replaced by

   $$\frac{}{x : A \vdash x : A} \qquad \text{and} \qquad \frac{}{\vdash () : 1}$$

   and where contexts are lists (and not sets)?

4. Let $\mathcal{C}$ be a cartesian closed category. We suppose fixed a function $[\![-]\!]$ which to every type A associates an objects $[\![A]\!]$ such that $[\![A \times B]\!] = [\![A]\!] \times [\![B]\!]$ and $[\![A \Rightarrow B]\!] = [\![B]\!]^{[\![A]\!]}$. Define the interpretation of a sequent

$$x_1 : A_1, \dots, x_n : A_n \vdash M : A$$

as a morphism

$$[\![M]\!] : [\![A_1]\!] \times \dots \times [\![A_n]\!] \to [\![A]\!]$$

such that this interpretation is invariant by β- and η-equivalences (i.e. if $M \equiv_{\beta\eta} N$ then $[\![M]\!] = [\![N]\!]$).

5. Conversely, explain how to build a category Λ whose objects are types and morphisms are simply typed λ-terms. Check that this category is cartesian closed.

3 Compact closed categories

A monoidal category $\mathcal{C}$ is *closed* when then functor $- \otimes B : \mathcal{C} \to \mathcal{C}$ admits a right adjoint.

1. Recall that we can define a notion of adjunction inside any 2-category. Recall also that a strict monoidal category is a 2-category with one 0-cell. The notion of adjunction recasted into strict monoidal categories gives rise to the notion of (right or left) *dual* to an object. Make this explicit, and generalize this to non-strict monoidal categories.

2. Show that two left (resp. right) duals for an object are canonically isomorphic. Show that if $\mathcal{C}$ is symmetric monoidal, a left and a right dual for an object are canonically isomorphic.

3. A *compact closed* category is a symmetric monoidal category in which every object A has a dual A^*. Show that **Vect**, restricted to finite-dimensional vector spaces, is compact closed.

4. Show that every compact closed category is monoidal closed using

$$A \multimap B \quad = \quad A^* \otimes B$$

5. Define the trace operator

$$\mathrm{tr} \quad : \quad A \multimap B \quad \to \quad 1$$