TD3 – Adjunctions, monads

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1 Monads generated by an adjunction

1. Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is left adjoint to a functor $G : \mathcal{D} \to \mathcal{C}$ iff there exists two natural transformations

$$\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F \qquad \text{and} \qquad \varepsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$$

respectively called the *unit* and *counit* of the adjunction, such that

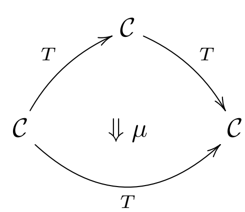
$$\varepsilon_F \cdot F\eta = \mathrm{id}_F \qquad \text{and} \qquad G\varepsilon \cdot \eta_G = \mathrm{id}_G \tag{1}$$

Describe the unit and counit corresponding the adjunctions studied in previous questions.

2. Show the above property.

3. Recall that a 2-category of categories, functors and natural transformations can be defined. What are the vertical and horizontal compositions in this category? What is the "exchange law" in a 2-category?

4. For every monad $T : \mathcal{C} \to \mathcal{C}$, the multiplication μ can be thus seen as a 2-cell



in this 2-category. By constructing the Poincaré dual of this diagram, we thus get a representation of the natural transformation μ using *string diagrams*. Similarly, give the string diagrammatic representation of the laws defining a monad as well as the laws (1).

5. Given an adjunction $(F, G, \eta, \varepsilon)$, show that the functor $G \circ F$ can be equipped with a structure of monad.

6. What are the monads associated the adjunction whose right adjoint is the forgetful functor from **pSet**/**Mon**/**Vect**/**Top** to **Set**?

7. Show that the forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ also admits a right adjoint.

8. [Optional] Show that if T is a monad on a category $\mathcal{C}$ then the category $\mathcal{C}$ is in adjunction with the category $\mathcal{C}_T$.

2 Monads in Rel

We define **Rel** as the 2-category whose 0-cells are sets, 1-cells $R : A \to B$ are relations $R \subseteq A \times B$, there is a unique 2-cell $\alpha : R \Rightarrow R' : A \to B$ whenever $R \subseteq R'$.

1. Recall both horizontal and vertical compositions in **Rel**.
2. Show that a left adjoint in **Rel** is a function.
3. What is a monad in **Rel**?

3 Monads in Haskell

Here is an excerpt of `http://www.haskell.org/haskellwiki/Monad`:

```
Monads can be viewed as a standard programming interface
to various data or control structures, which is captured
by the Monad class. All common monads are members of it:

class Monad m where
  (>>=) :: m a -> (a -> m b) -> m b
  return :: a -> m a

In addition to implementing the class functions, all
instances of Monad should obey the following equations:

return a >>= k  =  k a
m >>= return  =  m
m >>= (\x -> k x >>= h)  =  (m >>= k) >>= h
```

1. Show that this notion of monad is equivalent to the categorical definition of monads.
2. What does the `Maybe` monad defined below do?

   ```
   data Maybe a = Nothing | Just a

   instance Monad Maybe where
       return         = Just
       Nothing  >>= f = Nothing
       (Just x) >>= f = f x
   ```

3. What does the `List` monad defined below do?

   ```
   instance Monad [] where
       m >>= f  = concatMap f m
       return x = [x]
   ```