TD2 – Graphs, adjunctions, monads

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1 Graphs

1. Show that the category $\mathbf{Cat}(\mathbf{Gr}, \mathbf{Set})$ of functors and natural transformations from the category with two objects 0, 1 and four morphisms

$$\mathrm{id}_0 : 0 \to 0 \qquad \mathrm{id}_1 : 1 \to 1 \qquad s, t : 1 \to 0$$

to the category **Set** of sets and functions defines the category of graphs, which is usually denoted **Graph**.

2. Reformulate the definition of a category as a graph with some structure.

3. Explain that the category $\mathbf{Cat}(\mathbf{Gr}_2, \mathbf{Set})$ of functors from the category $\mathbf{Gr}_2$ with three objects 0, 1, 2 and nine morphisms

$$\mathrm{id}_0 : 0 \to 0 \qquad \mathrm{id}_1 : 1 \to 1 \qquad \mathrm{id}_2 : 2 \to 2 \qquad s_1, t_1 : 2 \to 1 \qquad s_0, t_0 : 1 \to 0 \qquad s, t : 2 \to 0$$

with

$$s_0 \circ s_1 = s_0 \circ t_1 = s \qquad \text{et} \qquad t_0 \circ s_1 = t_0 \circ t_1 = t$$

defines a category of 2-graphs and morphisms of 2-graphs.

4. Reformulate the definition of 2-categories using the notion of 2-graph.

2 Adjunctions between sets

We recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is *left adjoint* to a functor $G : \mathcal{D} \to \mathcal{C}$ if there is a natural bijection between $\mathcal{D}(FA, B)$ and $\mathcal{C}(A, GB)$.

1. Suppose given two functions $f : A \to B$ and $g : B \to A$ between sets A and B. Show that the two following properties are equivalent:
   (i) f and g are bijections and $f = g^{-1}$
   (ii) $\forall a \in A, \forall b \in B, \quad f(a) = b \quad \text{iff} \quad a = g(b)$

2. Conclude that an adjunction between two discrete categories is a bijection.

3 The exception monad

We write **pSet** for the category whose objects are *pointed sets*, i.e. pairs (A, a) where A is a set and $a \in A$, and morphisms $f : (A, a) \to (B, b)$ are functions such that $f(a) = b$. Here the distinguished element of the pointed set will be seen as a particular value indicating an error or an exception.

1. Describe the *forgetful functor* $U : \mathbf{pSet} \to \mathbf{Set}$ which to a pointed set associates the underlying set.

2. Construct a functor $F : \mathbf{Set} \to \mathbf{pSet}$ which is such that the sets $\mathrm{Hom}(FA, B)$ and $\mathrm{Hom}(A, UB)$ are isomorphic.

3. [Facultative] Show that the families of isomorphisms

$$\varphi_{A,B} : \mathrm{Hom}(FA, B) \to \mathrm{Hom}(A, UB) \qquad \text{and} \qquad \psi_{A,B} : \mathrm{Hom}(A, UB) \to \mathrm{Hom}(FA, B)$$

described in previous question are natural. By "$\varphi_{A,B}$ is *natural*", we mean here that for every morphisms $f : A \to A'$ in $\mathbf{Set}$ and $g : B \to B'$ in $\mathbf{pSet}$ the diagram

$$\begin{array}{ccc}
\mathrm{Hom}(FA', B) & \xrightarrow{\phi_{A',B}} & \mathrm{Hom}(A', UB) \\
{\scriptstyle g\circ-\circ Ff}\downarrow & & \downarrow{\scriptstyle Ug\circ-\circ f} \\
\mathrm{Hom}(FA, B') & \xrightarrow[\phi_{A,B'}]{} & \mathrm{Hom}(A, UB')
\end{array}$$

commutes (in $\mathbf{Set}$). Naturality of ψ is defined in a similar way.

4. We recall that a *monad* consists of an endofunctor $T : \mathcal{C} \to \mathcal{C}$ together with two natural transformations $\mu : T \circ T \Rightarrow T$ and $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow T$ such that the following diagrams commute:

$$\begin{array}{ccc}
T \circ T \circ T & \xRightarrow{T\mu} & T \circ T \\
{\scriptstyle \mu_T}\Downarrow & & \Downarrow{\scriptstyle \mu} \\
T \circ T & \xRightarrow[\mu]{} & T
\end{array}
\qquad\qquad
\begin{array}{ccccc}
T & \xRightarrow{\eta_T} & T \circ T & \xLeftarrow{T\eta} & T \\
& {\scriptstyle \mathrm{id}_T}\searrow & \Downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \mathrm{id}_T} & \\
& & T & &
\end{array}$$

Represent those diagrams using pasting diagrams in the 2-category $\mathbf{Cat}$. Represent those diagrams using string diagrams.

5. Describe a structure of monad on $U \circ F$.

6. Given $f : A \to B$ an OCaml function which might raise an unique exception e and $g : B \to C$ a function which might raise an unique exception e', construct a function corresponding to the composite of f and g which might raise a unique exception e''.

7. We write $\mathbf{Set}_T$ the category whose objects are the objects of $\mathbf{Set}$ and morphisms $f : A \to B$ in $\mathbf{Set}_T$ are morphisms $f : A \to TB$ in $\mathbf{Set}$. Compositions of two morphisms $f : A \to B$ and $g : B \to C$ in $\mathbf{Set}_T$ is defined by $g \circ f = \mu_C \circ Tg \circ f$ and identities are $\mathrm{id}_A = \eta_A$. Show that the axioms of categories are satisfied.

8. Give an explicit description of $\mathbf{Set}_T$.

9. A *non-deterministic function* is a function that might return a set of values instead of a single value. How could we could we similarly define a category of non-deterministic functions by a Kleisli construction?

4 Free category on a graph

1. Define the forgetful functor $U : \mathbf{Cat} \to \mathbf{Graph}$.

2. Show that this functor $F : \mathbf{Graph} \to \mathbf{Cat}$ admits a left adjoint.