TD8 – Limits, presheaf categories

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1 Limits

Suppose given a functor $S : \mathcal{D} \to \mathcal{C}$ and c and object of $\mathcal{C}$. An *universal arrow* from c to S is given by a pair (r, u) where r is an object of $\mathcal{D}$ and $u : c \to Sr$ is a morphism in $\mathcal{C}$ such that for every other such pair (d, f) (where d is an object of $\mathcal{C}$ and $f : c \to Sd$ is a morphism of $\mathcal{C}$), there exists a unique morphism $f' : r \to d$ of $\mathcal{D}$ such that $Sf' \circ u = f$.

1. Suppose that $U : \mathcal{D} \to \mathcal{C}$ is a functor admitting a left adjoint $F : \mathcal{C} \to \mathcal{D}$. Show that for every object X of $\mathcal{D}$, (FX, η_X) is a universal arrow from X to U.

Suppose given two categories $\mathcal{J}$ and $\mathcal{C}$. The *diagonal functor* $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$ is such that

- for every object $C \in \mathcal{C}$, $\Delta(C)$ sends every object of $\mathcal{J}$ to C and every morphism of $\mathcal{J}$ to id_C,
- for every morphism $f : C \to D \in \mathcal{C}$, $\Delta(f)$ is the natural transformation whose components are f.

The *limit* of a functor $F : \mathcal{J} \to \mathcal{C}$ is a co-universal arrow from Δ to F.

2. What is the limit of a functor F in the case where $\mathcal{J}$ is the terminal category.
3. Express the notions of product and fibred product in terms of limits.
4. Explain the dual notion of colimit.
5. Show that a category has pushouts when it has coproducts and coequalizers.

2 Presheaf categories as free cocompletions

We recall that the category of presheaves $\hat{\mathcal{C}}$ over a category $\mathcal{C}$ is the category of functors $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ and natural transformations between them.

1. Recall that graphs and simplicial sets are presheaf categories.
2. The Yoneda embedding $y : \mathcal{C} \to \hat{\mathcal{C}}$ is defined on objects $A \in \mathcal{C}$ by $yA = \mathcal{C}(-, A)$. Make completely explicit the definition of this functor.
3. [Yoneda lemma] Show that for every $A \in \mathcal{C}$ and $P \in \hat{\mathcal{C}}$, $\hat{\mathcal{C}}(yA, P) \cong P(A)$.
4. Deduce that the Yoneda embedding is full and faithful.
5. What is the Yoneda embedding of the objects in the case of graphs and simplicial sets?
6. Explain why every presheaf category is complete and cocomplete.
7. Describe a functor $I : \Delta \to \mathbf{Top}$ sending n to the canonical n-simplex.

8. Use this functor in order to build a *nerve* functor $N_I : \mathbf{Top} \to \hat{\Delta}$ associating to every topological space a simplicial set.

To any presheaf $P \in \hat{\mathcal{C}}$, we can associate a *category of elements* whose objects are pairs (A, a) with $A \in \mathcal{C}$ and $a \in P(A)$, and morphisms $f : (A, a) \to (B, b)$ are morphisms $f : A \to B$ of $\mathcal{C}$ such that $P(f)(b) = a$. We write $\pi_P : \mathrm{El}(P) \to \mathcal{C}$ for the first projection functor. We define the *geometric realization* functor by

$$R_I(P) \quad = \quad \mathrm{colim}(\mathrm{El}(P) \xrightarrow{\pi_P} \Delta \xrightarrow{I} \mathbf{Top})$$

8. Compute the geometric realization of a simple simplicial set ($y3$ for instance).

9. Show that R_I is left adjoint to N_I.

10. Notice that the above proofs could be generalized to any functor $I : \mathcal{C} \to \mathcal{D}$ and deduce that any presheaf $P \in \hat{\mathcal{C}}$ is canonically a colimit of representables:

$$P \quad = \quad \mathrm{colim}(\mathrm{El}(P) \xrightarrow{\pi_P} \mathcal{C} \xrightarrow{y} \hat{\mathcal{C}})$$

    We admit the following result: given a adjunction, the right adjoint is full and faithful if and only if the counit is an isomorphism.

11. Show that $\hat{\mathcal{C}}$ is the free cocompletion of $\mathcal{C}$: given a functor $F : \mathcal{C} \to \mathcal{D}$, there exists a unique cocontinuous functor $G : \hat{\mathcal{C}} \to \mathcal{D}$ such that $G \circ y = F$.