TD6 – Cartesian categories

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1 Cartesian categories

Suppose given a category $\mathcal{C}$. A *cartesian product* of two objects A and B is given by an object $A \times B$ together with two morphisms

$$\pi_1 : A \times B \to A \qquad \text{and} \qquad \pi_2 : A \times B \to B$$

such that for every object C and morphisms $f : C \to A$ and $g : C \to B$, there exists a unique morphism $h : C \to A \times B$ making the diagram

$$\begin{tikzcd}
A & & B \\
& A \times B \arrow[ul, "\pi_1"'] \arrow[ur, "\pi_2"] & \\
& C \arrow[uul, "f"] \arrow[uur, "g"'] \arrow[u, dotted, "h"] &
\end{tikzcd}$$

commute. We also recall that a *terminal object* in a category is an object 1 such that for every object A there exists a unique morphism $f_A : A \to 1$. A category is *cartesian* when it has finite products, i.e. has a terminal object and every pair of objects admits a product.

1. Suppose that $(E, \leq)$ is a poset. We associate to it category whose objects are elements of E and such that there exists a unique morphism between object a and b iff $a \leq b$. What is an initial object and a product in this category?
2. Show that the category **Set** of sets and functions is cartesian.
3. Show that two terminal objects in a category are necessarily isomorphic.
4. Similarly, show that the cartesian product of two objects is defined up to isomorphism.
5. Show that for every object A of a cartesian category, the objects $1 \times A$, A and $A \times 1$ are isomorphic.
6. Show that for every objects A and B, $A \times B$ and $B \times A$ are isomorphic.
7. Show that for every objects A, B and C, $(A \times B) \times C$ and $A \times (B \times C)$ are isomorphic.
8. The notion of *coproduct* is dual to the notion of product. Show that **Set** has all coproducts and an initial object.
9. Show that the category **Rel** of sets and relations is cartesian.
10. Show that the category **Cat** is cartesian.
11. Given a cartesian category $\mathcal{C}$, show that the cartesian product induces a functor $A, B \mapsto A \times B : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

12. Given a category $\mathcal{C}$, show that $\mathrm{Hom}_{\mathcal{C}}(-,-)$ induces a functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$.
13. Given an adjunction $F \dashv G : \mathcal{C} \to \mathcal{D}$, we have a natural bijection $\mathcal{D}(F-,-) \cong \mathcal{C}(-,G-)$. Elaborating on previous question, make the naturality condition explicit.

2 Cartesian closed categories

Recall that a category $\mathcal{C}$ is *cartesian closed* when it has finite products and for every object B of $\mathcal{C}$ the functor $- \times B : \mathcal{C} \to \mathcal{C}$ admits a right adjoint, written $(-)^B$, i.e. there exists a bijection $\mathcal{C}(A \times B, C) \cong \mathcal{C}(A, C^B)$ natural in A and C.

1. Show that the category **Set** is cartesian closed.
2. Describe the unit and the counit of the adjunction defining the closure. Check that the laws between the unit and counit in an adjunction are satisfied.
3. Show that the category **Cat** is cartesian closed.

3 Simply typed λ-calculus

Recall the syntax of λ-terms with products:

$$M \quad ::= \quad x \quad | \quad \lambda x.M \quad | \quad MM \quad | \quad (M,M) \quad | \quad \pi_1 \quad | \quad \pi_2 \quad | \quad ()$$

and the syntax of types:

$$A \quad ::= \quad a \quad | \quad A \times A \quad | \quad 1$$

The λ-terms will be considered modulo α-conversion (renaming of bound variables). The β-conversion rules are defined by

$$(\lambda x.M)N \equiv_\beta M[N/x] \qquad\qquad \pi_i(M_1, M_2) \equiv_\beta M_i$$

and those of η-conversion by

$$\lambda x.Mx \equiv_\eta M \qquad\qquad (\pi_1 M, \pi_2 M) \equiv_\eta M \qquad\qquad M \equiv_\eta () \text{ if } M \text{ has type } 1$$

1. Recall the typing rules.
2. Define the substitution operation $M[N/x]$ by induction on the structure of the term M. Show that typing is preserved under β-reduction.
3. We want to make explicit context manipulations. Which rules do we have to add if we want to obtain an equivalent deduction system where the axiom and unit rule have been replaced by

   $$\frac{}{x : A \vdash x : A} \qquad \text{and} \qquad \frac{}{\vdash () : 1}$$

   and where contexts are lists (and not sets)?
4. Let $\mathcal{C}$ be a cartesian closed category. We suppose fixed a function $[\![-]\!]$ which to every type A associates an objects $[\![A]\!]$ such that $[\![A \times B]\!] = [\![A]\!] \times [\![B]\!]$ and $[\![A \Rightarrow B]\!] = [\![B]\!]^{[\![A]\!]}$. Define the interpretation of a sequent

   $$x_1 : A_1, \ldots, x_n : A_n \vdash M : A$$

   as a morphism

   $$[\![M]\!] : [\![A_1]\!] \times \ldots \times [\![A_n]\!] \to [\![A]\!]$$

   such that this interpretation is invariant by β- and η-equivalences (i.e. if $M \equiv_{\beta\eta} N$ then $[\![M]\!] = [\![N]\!]$).
5. Conversely, explain how to build a category Λ whose objects are types and morphisms are simply typed λ-terms. Check that this category is cartesian closed.