TD4 – Rewriting

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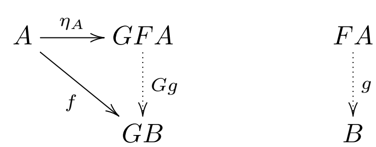
1 String rewriting systems

1. Given two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and a natural transformation $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$, show that F is left adjoint to G with η as unit if and only if for every objects $A \in \mathcal{C}$, $B \in \mathcal{D}$ and morphism $f : A \to GB$ in $\mathcal{D}$, there exists a unique morphism $g : FA \to B$ such that the following diagram commutes:



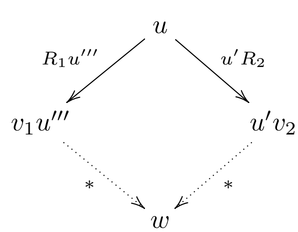
2. Recall the definition of the category **Mon** of monoids and construct a left adjoint to the forgetful functor U : **Mon** $\to$ **Set**. What does the above theorem says in the case of this adjunction?

A *string rewriting system* consists of an *alphabet* Σ and a set of *rules* of the form $R : u \to v$ where $u, v \in \Sigma^*$ are words over Σ. On says that a word w *rewrites* to a word w' when there exists $w_1, w_2 \in \Sigma^*$ and a rule $R : u \to v$ such that $w = w_1 u w_2$ and $w' = w_1 v w_2$, in which case we write $w_1 R w_2 : w \to w'$.

3. We consider the rewriting system with $\Sigma = \{a, b\}$ and one rule $R : aa \Rightarrow 1$. Use previous question to construct a morphism $\phi : \Sigma^* \to \mathbb{N}$ which "counts" the number of a's in a word.

4. Deduce that the rewriting system is terminating.

A *critical pair* is a pair of rules $R_1 : u_1 \to v_1$ and $R_2 : u_2 \to v_2$ together with words u', u'' and u''' such that $u = u'u''u'''$, $u'' \neq \varepsilon$, $u'u'' = u_1$ and $u''u''' = u_2$. Such a critical pair is *joinable* when there exists a word w such that



5. Show that a string rewriting system is confluent if and only if all its critical pairs are joinable.

6. Show that the above string rewriting system is convergent.

2 Presentation of monoids

1. Show that the string rewriting system with $\Sigma = \{a, b\}$ and rule $R_1 : ba \to ab$ and $R_2 : bb \to 1$ is confluent.

A partial order is *well-founded* when it has no infinite sequence of strictly decreasing elements.

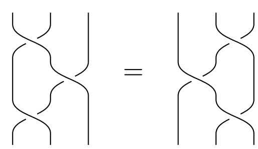
2. Given two well-founded posets, define the lexicographic order on their product and show that it is well-founded.

3. Show that the above rewriting system is terminating.

A *presentation* of a monoid M is a pair $\langle G \mid R \rangle$ where G is a set (of *generators*) and $R \subseteq G^* \times G^*$ of *relations*, such that $M \cong (R/\approx)$ where $\approx$ is the smallest congruent (wrt concatenation) containing the relations (the isomorphism is an isomorphism of monoids).

4. Show that $\mathbb{N} \times \mathbb{N}/2\mathbb{N}$ admits the presentation $\langle a, b \mid ba = ab, bb = 1 \rangle$.

The *braid group* B_3 on three strands satisfies the following law:



5. [Informal] Provide a presentation of the monoid B_3^+ of braids on 3 strands. Extend it to get a presentation of the group B_3. Provide a presentation of B_n (braids on n strands).

6. Show in B_6: $a_3a_1a_5a_2a_3a_3 = a_1a_5a_2a_3a_2a_3$.

7. The group of symmetries S_n can be obtained from B_n^+ by quotienting so that the twist is its own inverse. Starting from this provide a presentation of S_n and show that S_4 is actually admits this presentation.

3 Confluence of λ-calculus by finite developments

In order to show that λ-calculus in confluent, we are going to show that if we choose a set of redexes in a λ-term then reducing only those (or their residuals) is convergent. We define the $\underline{\lambda}$-calculus by

$$M \quad ::= \quad x \quad | \quad MM \quad | \quad \lambda x.M \quad | \quad \underline{(\lambda x.M)M}$$

together with the $\underline{\beta}$-reduction rule $\underline{(\lambda x.M)N} \to M[N/x]$. Notice that in this calculus $\underline{\lambda x.M}$ is not a term and $(\lambda x.M)N$ is not a redex.

1. Show that $\underline{\beta}$-reduction is terminating.

2. Show that $\underline{\beta}$-reduction is strongly normalizing. What do normal forms look like?

Notice that any λ-term can be seen as a $\underline{\lambda}$-term. Conversely, we define an operation of "erasing underlines" E from $\underline{\lambda}$-terms to λ-terms by

$$E(x) = x \qquad E(MN) = E(M)E(N) \qquad E(\lambda x.M) = \lambda x.E(M) \qquad E(\underline{(\lambda x.M)N}) = (\lambda x.E(M))E(N)$$

Given two λ-terms M and N, we write $M \to_1 N$ whenever there exists a $\underline{\lambda}$-term M' such that $M = E(M')$ and N is the normal form of M'.

3. Show that the reduction $\to_1$ is strongly normalizing.

4. Deduce that β-reduction is convergent.