

# TD3 – Adjunctions, monads

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October 18, 2012

## 1 Monads generated by an adjunction

1. Recall that a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is left adjoint to a functor  $G : \mathcal{D} \rightarrow \mathcal{C}$  iff there exists two natural transformations

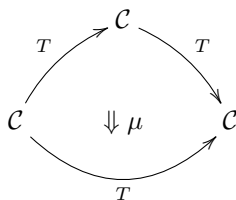
$$\eta : \text{id}_{\mathcal{C}} \rightarrow G \circ F \quad \text{and} \quad \varepsilon : F \circ G \rightarrow \text{id}_{\mathcal{D}}$$

respectively called the *unit* and *counit* of the adjunction, such that

$$\varepsilon_F \cdot F\eta = \text{id}_F \quad \text{and} \quad G\varepsilon \cdot \eta_G = \text{id}_G \tag{1}$$

Describe the unit and counit corresponding the adjunctions studied in previous questions.

2. Show the above property.
3. Recall that a 2-category of categories, functors and natural transformations can be defined. What are the vertical and horizontal compositions in this category? What is the “exchange law” in a 2-category?
4. For every monad  $T : \mathcal{C} \rightarrow \mathcal{C}$ , the multiplication  $\mu$  can be thus seen as a 2-cell



in this 2-category. By constructing the Poincaré dual of this diagram, we thus get a representation of the natural transformation  $\mu$  using *string diagrams*. Similarly, give the string diagrammatic representation of the laws defining a monad as well as the laws (1).

5. Given an adjunction  $(F, G, \eta, \varepsilon)$ , show that the functor  $G \circ F$  can be equipped with a structure of monad.
6. What are the monads associated the adjunction whose right adjoint is the forgetful functor from **pSet/Mon/Vect/Top** to **Set**?
7. Show that the forgetful functor  $U : \mathbf{Top} \rightarrow \mathbf{Set}$  also admits a right adjoint.
8. [Optional] Show that if  $T$  is a monad on a category  $\mathcal{C}$  then the category  $\mathcal{C}$  is in adjunction with the category  $\mathcal{C}_T$ .

## 2 Monads in Rel

We define **Rel** as the 2-category whose 0-cells are sets, 1-cells  $R : A \rightarrow B$  are relations  $R \subseteq A \times B$ , there is a unique 2-cell  $\alpha : R \Rightarrow R' : A \rightarrow B$  whenever  $R \subseteq R'$ .

1. Recall both horizontal and vertical compositions in **Rel**.
2. Show that a left adjoint in **Rel** is a function.
3. What is a monad in **Rel**?

## 3 Monads in Haskell

Here is an excerpt of <http://www.haskell.org/haskellwiki/Monad>:

```
Monads can be viewed as a standard programming interface
to various data or control structures, which is captured
by the Monad class. All common monads are members of it:
```

```
class Monad m where
  (>>=) :: m a -> (a -> m b) -> m b
  return :: a -> m a
```

In addition to implementing the class functions, all instances of Monad should obey the following equations:

```
return a >>= k = k a
m >>= return = m
m >>= (\x -> k x >>= h) = (m >>= k) >>= h
```

1. Show that this notion of monad is equivalent to the categorical definition of monads.
2. What does the Maybe monad defined below do?

```
data Maybe a = Nothing | Just a
```

```
instance Monad Maybe where
  return = Just
  Nothing >>= f = Nothing
  (Just x) >>= f = f x
```

3. What does the List monad defined below do?

```
instance Monad [] where
  m >>= f = concatMap f m
  return x = [x]
```