

TD4 – λ -calculus

Samuel Mimram

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1 Reduction graphs

The *reduction graph* of a λ -term M is the graph, whose vertices are λ -terms, defined as the smallest graph such that M is a vertex and there is an arrow between two vertices M and M' whenever $M \rightarrow_{\beta} M'$.

1. Write the respective reduction graphs of

$$(\lambda x.xx)(\lambda y.y)z \quad \text{and} \quad (\lambda xy.x)((\lambda x.xx)(\lambda xy.xy))$$

2. Can a reduction graph have loops?

2 Booleans

We encode the booleans \top and \perp into λ -terms respectively as

$$\llbracket \top \rrbracket = \text{true} = \lambda x.\lambda y.x \quad \text{and} \quad \llbracket \perp \rrbracket = \text{false} = \lambda x.\lambda y.y$$

1. Define λ -terms *and*, *or* and *not* such that for every booleans b and b' ,

$$\text{and}\llbracket b \rrbracket\llbracket b' \rrbracket \rightarrow_{\beta} \llbracket b \wedge b' \rrbracket \quad \text{or}\llbracket b \rrbracket\llbracket b' \rrbracket \rightarrow_{\beta} \llbracket b \vee b' \rrbracket \quad \text{not}\llbracket b \rrbracket \rightarrow_{\beta} \llbracket \neg b \rrbracket$$

2. Define a λ -term *if* such that

$$\text{if}\llbracket \top \rrbracket MN \xrightarrow{*}_{\beta} M \quad \text{and} \quad \text{if}\llbracket \perp \rrbracket MN \xrightarrow{*}_{\beta} N$$

3 Church numerals

The Church encoding of integers n in λ -calculus is

$$\llbracket n \rrbracket = \lambda fx. \underbrace{f(f \dots (fx))}_{n \text{ times}}$$

1. Define the interpretation of the successor, test to zero, addition, multiplication and exponential functions.
2. We assume that the predecessor function can be coded¹. Give a recursive definition of the factorial function in λ -calculus.
3. We define $\theta = \lambda gh.h(gh)$ and $\Theta = \theta\theta$. Show that Θ is a *fixpoint operator*, i.e. $\Theta f \xrightarrow{*}_{\beta} f(\Theta f)$.
4. Use the preceding combinator to define the interpretation of the factorial function in λ -calculus.

¹by $\lambda nfx.(\lambda gh.h(gh))(\lambda u.x)(\lambda u.u)$

4 Weak normalization of the λ -calculus

An *abstract rewriting system* (ARS) is a graph whose vertices are called *terms* and whose edges are called *rewriting rules*. We often write $x \rightarrow y$ when there exists an edge from x to y and $x \xrightarrow{*} y$ when there exists a directed path from x to y (in the latest case, we say that x rewrites to y). An ARS is

- *locally confluent* when $y_1 \leftarrow x \rightarrow y_2$ implies that there exists z such that $y_1 \xrightarrow{*} z \xleftarrow{*} y_2$,
- *confluent* when $y_1 \xleftarrow{*} x \xrightarrow{*} y_2$ implies that there exists z such that $y_1 \xrightarrow{*} z \xleftarrow{*} y_2$,
- *strongly confluent* when $y_1 \leftarrow x \rightarrow y_2$ implies that there exists z such that $y_1 \rightarrow z \leftarrow y_2$.

1. Which properties imply another? Give counter-examples for implications which fail.
2. A *normal form* is a term x such that there is no y for which $x \rightarrow y$. Show that in a confluent rewriting system a term reduces to at most one normal form.
3. [Newman's lemma] An ARS is *terminating* if it does not contain any infinite path. Show that an ARS which is terminating and locally confluent is confluent. What can you say about normal forms in such a rewriting system?
4. Describe the abstract rewriting system of λ -terms with β -reduction.
5. A λ -term is *strongly terminating* when it can only be reduced a finite number of times, *divergent* when it does not reduce to a normal form and *weakly terminating* when it can reduce to a normal form. Give example of λ -terms with such properties.
6. The *parallel reduction* $M \Rightarrow N$ on λ -terms is defined by:
 - $M \Rightarrow M$
 - $M \Rightarrow M'$ and $N \Rightarrow N'$ implies $MN \Rightarrow M'N'$
 - $M \Rightarrow M'$ implies $\lambda x.M \Rightarrow \lambda x.M'$
 - $M \Rightarrow M'$ and $N \Rightarrow N'$ implies $(\lambda x.M)N \Rightarrow M'[N'/x]$

Show that \Rightarrow is strongly confluent.

7. Show that $\rightarrow_{\beta} \subseteq \Rightarrow \subseteq \rightarrow_{\beta}^*$. Provide counter-examples showing that these inclusions are strict.
8. Conclude that \rightarrow_{β} is confluent.