TD6 – Adjunctions and monads

1 Monade d'exception

On note **pEns** le catégorie dont les objets sont les *ensembles pointés*, c'est-à-dire les paires (A, a) où A est un ensemble et $a \in A$, et dont les morphismes $f : (A, a) \to (B, b)$ sont les fonctions $f : A \to B$ telles que $f(a) = b$. Ici, l'élément distingué d'un ensemble pointé sera vu comme une valeur particulière indiquant une erreur.

1. Décrivez le *foncteur d'oubli* U : **pEns** $\to$ **Ens** qui à un ensemble pointé associe l'ensemble sous-jacent.
2. Construisez un foncteur F : **Ens** $\to$ **pEns** qui soit tel que les ensembles $\mathrm{Hom}(FA, B)$ et $\mathrm{Hom}(A, UB)$ soient isomorphes.
3. [Facultatif] Montrez que les familles d'isomorphismes

$$\varphi_{A,B} : \mathrm{Hom}(FA, B) \to \mathrm{Hom}(A, UB) \quad \text{et} \quad \psi_{A,B} : \mathrm{Hom}(A, UB) \to \mathrm{Hom}(FA, B)$$

que vous avez décrites à la question précédente sont naturels. Par « $\varphi_{A,B}$ est *naturelle* », on entend que pour tous morphismes $f : A \to A'$ dans **Ens** et $g : B \to B'$ and **pEns** le diagramme

$$\begin{array}{ccc} \mathrm{Hom}(FA', B) & \xrightarrow{\phi_{A',B}} & \mathrm{Hom}(A', UB) \\ {\scriptstyle g\circ - \circ Ff}\downarrow & & \downarrow{\scriptstyle Ug\circ - \circ f} \\ \mathrm{Hom}(FA, B') & \xrightarrow[\phi_{A,B'}]{} & \mathrm{Hom}(A, UB') \end{array}$$

commute (dans **Ens**). La naturalité de ψ étant définie de façon similaire.

On appelle *adjonction* une telle paire de foncteurs $U : \mathcal{C} \to \mathcal{D}$ et $F : \mathcal{D} \to \mathcal{C}$ telle qu'il existe une bijection naturelle entre les ensembles $\mathrm{Hom}(FA, B)$ et $\mathrm{Hom}(A, UB)$, ce que l'on écrit parfois

$$\frac{FA \to B}{A \to UB}$$

Le foncteur F est alors appelé *adjoint à gauche de* U, ce que l'on note $F \dashv U$.

4. Rappelez la structure de monade sur le foncteur $U \circ F$.

2 Non-determinism monad

1. We write **Mon** for the category of monoids. Describe the functor U : **Mon** $\to$ **Ens** which sends a monoid to its underlying set. The functor U is often called a *forgetful functor* because it "forgets" about the structure of monoid on a set.
2. Give an explicit description of the monoid freely generated by a set.
3. Construct a functor F : **Ens** $\to$ **Mon** which sends a set on the monoid it freely generates.
4. Show that F is left adjoint to U.
5. Define a structure of monad on the functor $U \circ F$: **Ens** $\to$ **Ens**.
6. Similarly define a monad T : **Ens** $\to$ **Ens** from an adjunction between **Set** and the category **CMon** of commutative monoids.
7. Describe the Kleisli category $\mathbf{Ens}_T$ and explain why we can see its morphisms as non-deterministic programs.
8. Other variant : construct similarly the powerset monad on **Set** which to every set associates the set of its subsets, and give a direct description of the associated Kleisli category.

3 Free category on a graph

A *graph* is defined as a diagram $V \underset{t}{\overset{s}{\leftleftarrows}} E$ in **Set**.

1. Define the notion of morphism of graph. We write **Graph** for the category thus constructed.
2. Define the forgetful functor $U : \mathbf{Cat} \to \mathbf{Graph}$.
3. Show that this functor $F : \mathbf{Graph} \to \mathbf{Cat}$ admits a left adjoint.

4 Terminal objects and products by adjunctions

1. Show that the category **Cat** has a terminal object **1**.
2. Given a category $\mathcal{C}$, describe the *terminal functor* $T : \mathbf{Cat} \to \mathbf{1}$.
3. Given a category $\mathcal{C}$, show that the terminal functor $T : \mathcal{C} \to \mathbf{1}$ has a right (resp. left) adjoint iff the category $\mathcal{C}$ admits a terminal (resp. initial) object.
4. Given a category $\mathcal{C}$, describe the *diagonal functor* $D : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ and show that the category $\mathcal{C}$ admins cartesian products (resp. coproducts) iff the diagonal functor admits a right (resp. left) adjoint.

5 Monads generated by an adjunction

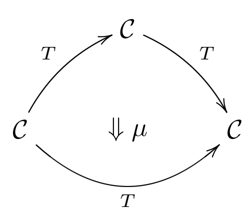
1. Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is left adjoint to a functor $G : \mathcal{D} \to \mathcal{C}$ iff there exists two natural transformations

   $$\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F \qquad \text{et} \qquad \varepsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$$

   respectively called the *unit* and *counit* of the adjunction, such that

   $$\varepsilon_F \cdot F\eta = \mathrm{id}_F \qquad \text{et} \qquad G\varepsilon \cdot \eta_G = \mathrm{id}_G \tag{1}$$

   Describe the unit and counit corresponding the adjunctions studied in previous questions.
2. Recall that a 2-category of categories, functors and natural transformations can be defined. What are the vertical and horizontal compositions in this category ? What is the “exchange law” in a 2-category ?
3. For every monad $T : \mathcal{C} \to \mathcal{C}$, the multiplication μ can be thus seen as a 2-cell

   in this 2-category. By constructing the Poincaré dual of this diagram, we thus get a representation of the natural transformation μ using *string diagrams*. Similarly, give the string diagrammatic representation of the laws defining a monad as well as the laws (1).
4. Given an adjunction $(F, G, \eta, \varepsilon)$, show that the functor $G \circ F$ can be equipped with a structure of monad.
5. [Optional] Show the property mentioned in question 1.
6. [Optional] Show that if T is a monad on a category $\mathcal{C}$ then the category $\mathcal{C}$ is in adjunction with the category $\mathcal{C}_T$.