TD6 – Semantics of fixpoint

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1 PCF

PCF (*Programming language for Computable Functions*) is a language constructed from λ-calculus by adding constants for cartesian products, integers, successor, predecessor, test to zero and fixpoint.

1. Define the syntax for PCF terms.
2. Define reduction rules for PCF terms.
3. The syntax for types is

   $$A \quad ::= \quad \texttt{nat} \quad | \quad A \Rightarrow A \quad | \quad A \times A$$

   Define a typing system for terms.
4. Program the addition of two integers.
5. Give an interpretation of types into **Set**.
6. Give an interpretation of sequents (excepting for fixpoint) and show that it is preserved by reduction.
7. Explain why we cannot interpret fixpoints.

2 Interpreting fixpoint

Suppose given a poset $(D, \leq)$. A relation $x \leq y$ should be thought as "y contains more information than x". We recall that a *chain* is a totally ordered subset of D, and an ω-chain is a chain isomorphic to the poset $\mathbb{N}$. A poset is an *ω-cpo* (= *complete partial order*) if every chain has a supremum and has a smallest element $\bot$. A function $f : (D, \leq_D) \to (E, \leq_E)$ is *continuous* if for every chain X, $f(\sup X) = \sup f(X)$.

1. Show that every continuous function is increasing.
2. Suppose that $f : (D, \leq) \to (D, \leq)$ is a continuous function where $(D, \leq)$ is an ω-cpo. A post-fixpoint of a function is an element x such that $x \leq f(x)$. Show that $X_x = \{f^n(x) \mid n \in \mathbb{N}\}$ is a chain. What can you say about $\sup X_x$? Show that f admits a smallest fixpoint.
3. Given two ω-cpos $(D_1, \leq_1)$ and $(D_2, \leq_2)$, define a structure of ω-cpo on $D_1 \times D_2$ and $D_1 \Rightarrow D_2$.
4. Show that the function which to a continuous function $f : (D, \leq) \to (D, \leq)$ associates its smallest fixpoint in $(D, \leq)$ is continuous.
5. Show that the category of ω-cpos and continuous functions is cartesian closed by defining the unit and co-unit for the adjunction defining the closure (and showing that they are continuous!).
6. Give an interpretation of PCF in the category of ω-cpos.
7. Compute the interpretation of addition.
8. *A confluence result.* Show that if a term reduces to integers $\underline{m}$ and $\underline{n}$ then $m = n$.

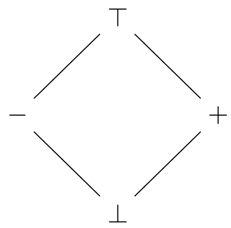
3 Towards abstract interpretation

Recall that a *complete lattice* $(D, \leq)$ is a poset in which every subset X of D admits a supremum and an infimum.

1. Show that an increasing function $f : (D, \leq) \to (D, \leq)$, where $(D, \leq)$ is a complete lattice, admits a smallest and a greatest fixpoint (in fact, the set of fixpoints can be shown to be itself a complete lattice).

2. Recall how a poset may be seen as a category. What is a functor between two poset categories? An adjunction? What is the monad generated by such an adjunction? What are initial and terminal objects in a poset category? What are products and sums?

3. Consider the posets $(\mathcal{P}(\mathbb{Z}), \subseteq)$ and $(S, \leq)$ defined as



   Show that they are complete lattices and define a Galois connection between them.

4. Consider the following program written in pseudo-code

```
let f x =
  n = 100;
  while (n != 0)
  {
    n--;
    x++;
  };
  return x;
```

   Write an equivalent program in PCF.

5. Define a semantics of PCF in the category of complete lattices and increasing functions in which the interpretation of `nat` is $(\mathcal{P}(\mathbb{Z}), \subseteq)$.

6. Use this semantics to show that `f 5` is positive.

7. Given a Galois connection $\alpha \dashv \gamma : (D, \leq_D) \to (E, \leq_E)$ between complete lattices, an *abstraction* of an increasing function $f : (D, \leq_D) \to (D, \leq_D)$ is an increasing function $f^\sharp : (E, \leq_E) \to (E, \leq_E)$ such that for every $x \in D$, $f(x) \leq \gamma \circ f^\sharp \circ \alpha(x)$. How can you construct an abstraction for every function using the Galois connection? Show that the abstraction thus constructed is the best possible one.

8. Using the previous Galois connection, construct a semantics of PCF in the category of complete lattices and increasing functions in which the interpretation of `nat` is $(S, \leq)$.

9. Use this semantics to show that `f 5` is positive.

10. How can you refine this semantics to take null functions in account? To have intervals of integers?