# TD5 – Cartesian closed categories

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## 1 Cartesian closed categories

Recall that a category $\mathcal{C}$ is *cartesian closed* when it has finite products and for every object $B$ of $\mathcal{C}$ the functor $- \times B : \mathcal{C} \to \mathcal{C}$ admits a right adjoint, written $(-)^B$, i.e. there exists a bijection $\mathcal{C}(A \times B, C) \cong \mathcal{C}(A, C^B)$ natural in $A$ and $C$.

1. Show that the category **Set** is cartesian closed.
2. Describe the unit and the counit of the adjunction defining the closure. Check that the laws between the unit and counit in an adjunction are satisfied.
3. Show that the category **Cat** is cartesian closed.

## 2 Simply typed $\lambda$-calculus

Recall the syntax of $\lambda$-terms with products:

$$M \quad ::= \quad x \quad | \quad \lambda x.M \quad | \quad MM \quad | \quad (M, M) \quad | \quad \pi_1 \quad | \quad \pi_2 \quad | \quad ()$$

and the syntax of types:

$$A \quad ::= \quad a \quad | \quad A \times A \quad | \quad 1$$

The $\lambda$-terms will be considered modulo $\alpha$-conversion (renaming of bound variables). The typing rules for $\lambda$-terms are:

$$\frac{}{\Gamma, x : A \vdash x : A} \qquad \frac{}{\Gamma \vdash () : 1}$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \Rightarrow B} \qquad \frac{\Gamma \vdash M : A \Rightarrow B \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$$

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : B}{\Gamma \vdash (M, N) : A \times B} \qquad \frac{\Gamma \vdash M : A_1 \times A_2}{\Gamma \vdash \pi_i M : A_i}$$

where the contexts $\Gamma$ are sets. The $\beta$-conversion rules are defined by

$$(\lambda x.M)N \equiv_\beta M[N/x] \qquad \pi_i(M_1, M_2) \equiv_\beta M_i$$

and those of $\eta$-conversion by

$$\lambda x.Mx \equiv_\eta M \qquad (\pi_1 M, \pi_2 M) \equiv_\eta M \qquad M \equiv_\eta () \text{ if } M \text{ has type } 1$$

1. Define the substitution operation $M[N/x]$ by induction on the structure of the term $M$.
2. Show that typing is preserved under $\beta$-reduction.
3. We want to make explicit context manipulations. Which rules do we have to add if we want to obtain an equivalent deduction system where the axiom and unit rule have been replaced by

   $$\frac{}{x : A \vdash x : A} \qquad \text{and} \qquad \frac{}{\vdash () : 1}$$

   and where contexts are lists (and not sets)?

4. Let $\mathcal{C}$ be a cartesian closed category. We suppose fixed a function $[\![-]\!]$ which to every type $A$ associates an objects $[\![A]\!]$ such that $[\![A \times B]\!] = [\![A]\!] \times [\![B]\!]$ and $[\![A \Rightarrow B]\!] = [\![B]\!]^{[\![A]\!]}$. Define the interpretation of a sequent

$$x_1 : A_1, \ldots, x_n : A_n \vdash M : A$$

as a morphism

$$[\![M]\!] : [\![A_1]\!] \times \ldots \times [\![A_n]\!] \to [\![A]\!]$$

such that this interpretation is invariant by $\beta$- and $\eta$-equivalences (i.e. if $M \equiv_{\beta\eta} N$ then $[\![M]\!] = [\![N]\!]$).

5. Conversely, explain how to build a category $\Lambda$ whose objects are types and morphisms are simply typed $\lambda$-terms. Check that this category is cartesian closed.

# 3 Parameter theorem

1. Given a category $\mathcal{C}$, we write $\mathcal{C}^{\mathrm{op}}$ for the category obtained from $\mathcal{C}$ by "reversing arrows", i.e. $\mathcal{C}^{\mathrm{op}}(A, B) \cong \mathcal{C}(B, A)$. Explain how the operation $\mathrm{Hom}(-,-)$ can be extended into a functor $\mathrm{Hom} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$.

2. Suppose that $F : \mathcal{C} \to \mathcal{D}$ is a functor which admits a right adjoint $G : \mathcal{D} \to \mathcal{C}$. Write the axiom that a natural transformation $\phi$ between functors $\mathrm{Hom}(F-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ and $\mathrm{Hom}(-,G-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ has to satisfy.

3. *Parameter theorem.* In the following, we admit the following theorem: if $\mathcal{C}$ is a cartesian closed category, the family of functors $(-)^B$ indexed by $B$ defines a unique functor $(-)^- : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ such that the bijections $\mathcal{C}(A \times B, C) \cong \mathcal{C}(A, C^B)$ are natural in $A$, $B$ and $C$. Write the naturality axiom.