TD4 – λ-calculus

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1 Reduction graphs

The *reduction graph* of a λ-term M is the graph, whose vertices are λ-terms, defined as the smallest graph such that M is a vertex and there is an arrow between two vertices M and M' whenever $M \to_\beta M'$.

1. Write the respective reduction graphs of

$$(\lambda x.xx)(\lambda y.y)z \qquad \text{and} \qquad (\lambda xy.x)((\lambda x.xx)(\lambda xy.xy))$$

2. Can a reduction graph have loops?

2 Booleans

We encode the booleans $\top$ and $\bot$ into λ-terms respectively as

$$[\![\top]\!] = true = \lambda x.\lambda y.x \qquad \text{and} \qquad [\![\bot]\!] = false = \lambda x.\lambda y.y$$

1. Define λ-terms *and*, *or* and *not* such that for every booleans b and b',

$$and[\![b]\!][\![b']\!] \to_\beta [\![b \wedge b']\!] \qquad or[\![b]\!][\![b']\!] \to_\beta [\![b \vee b']\!] \qquad not[\![b]\!] \to_\beta [\![\neg b]\!]$$

2. Define a λ-term if such that

$$if[\![\top]\!]MN \xrightarrow{*}_\beta M \qquad \text{and} \qquad if[\![\bot]\!]MN \xrightarrow{*}_\beta N$$

3 Church numerals

The Church encoding of integers n in λ-calculus is

$$[\![n]\!] = \lambda fx.\underbrace{f(f\ldots(f}_{n \text{ times}}\,x))$$

1. Define the interpretation of the successor, test to zero, addition, multiplication and exponential functions.
2. We assume that the predecessor function can be coded[1]. Give a recursive definition of the factorial function in λ-calculus.
3. We define $\theta = \lambda gh.h(ggh)$ and $\Theta = \theta\theta$. Show that Θ is a *fixpoint operator*, i.e. $\Theta f \xrightarrow{*}_\beta f(\Theta f)$.
4. Use the preceding combinator to define the interpretation of the factorial function in λ-calculus.

[1] by $\lambda nfx.(\lambda gh.h(gf))(\lambda u.x)(\lambda u.u)$

4 Weak normalization of the λ-calculus

An *abstract rewriting system* (ARS) is a graph whose vertices are called *terms* and whose edges are called *rewriting rules*. We often write $x \to y$ when there exists an edge from x to y and $x \overset{*}{\to} y$ when there exists a directed path from x to y (in the latest case, we say that x rewrites to y). An ARS is

- *locally confluent* when $y_1 \leftarrow x \to y_2$ implies that there exists z such that $y_1 \overset{*}{\to} z \overset{*}{\leftarrow} y_2$,
- *confluent* when $y_1 \overset{*}{\leftarrow} x \overset{*}{\to} y_2$ implies that there exists z such that $y_1 \overset{*}{\to} z \overset{*}{\leftarrow} y_2$,
- *strongly confluent* when $y_1 \leftarrow x \to y_2$ implies that there exists z such that $y_1 \to z \leftarrow y_2$.

1. Which properties imply another? Give counter-examples for implications which fail.
2. A *normal form* is a term x such that there is no y for which $x \to y$. Show that in a confluent rewriting system a term reduces to at most one normal form.
3. [Newman's lemma] An ARS is *terminating* if it does not contain any infinite path. Show that an ARS which is terminating and locally confluent is confluent. What can you say about normal forms in such a rewriting system?
4. Describe the abstract rewriting system of λ-terms with β-reduction.
5. A λ-term is *strongly terminating* when it can only be reduced a finite number of times, *divergent* when it does not reduce to a normal form and *weakly terminating* when it can reduce to a normal form. Give example of λ-terms with such properties.
6. The *parallel reduction* $M \Rightarrow N$ on λ-terms is defined by:
   - $M \Rightarrow M$
   - $M \Rightarrow M'$ and $N \Rightarrow N'$ implies $MN \Rightarrow M'N'$
   - $M \Rightarrow M'$ implies $\lambda x.M \Rightarrow \lambda x.M'$
   - $M \Rightarrow M'$ and $N \Rightarrow N'$ implies $(\lambda x.M)N \Rightarrow M'[N'/x]$

   Show that $\Rightarrow$ is strongly confluent.
7. Show that $\to_\beta \subseteq \Rightarrow \subseteq \to_\beta^*$. Provide counter-examples showing that these inclusions are strict.
8. Conclude that $\to_\beta$ is confluent.