CSC_51051_EP: Homotopy types

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École polytechnique

Part I

Equality

Identity types

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It is of course possible to directly give the rules satisfied by those types.

Identity types

Note that definitional equality implies propositional equality: the rule

$$\frac{\Gamma \vdash t = u : A}{\Gamma \vdash \mathsf{refl} : \mathsf{Id}_A(t, u)}$$

is admissible.

Rules for identity types

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For this reason, it is usually not taken in account.

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It is not clear at all that this is an equivalence relation, but we will see that it is the case.

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Part II

The axioms K and UIP

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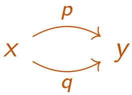

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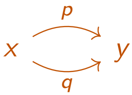


```
K-UIP : K → UIP
K-UIP K p q = loop-eq p q (K (λ r → r ≡ refl) refl (trans (sym p) q))
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So, case settled?

Part III

Types as spaces

With or without K?

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but this does not type because `p` is of type `x ≡ y`.

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It makes your life easier, but if you have too much of it you run into problems.

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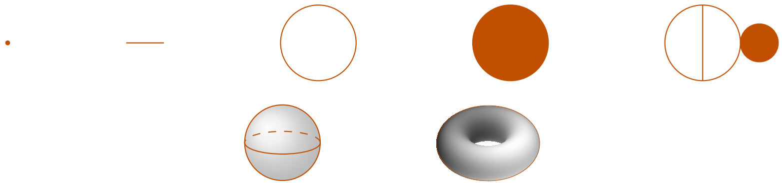
∞. in logic without UIP: a type is a **space**

Spaces

We will not precisely define what a space is, but you can think of it as

- a topological space, or
- something built up by gluing polyhedra (in arbitrary dimensions)



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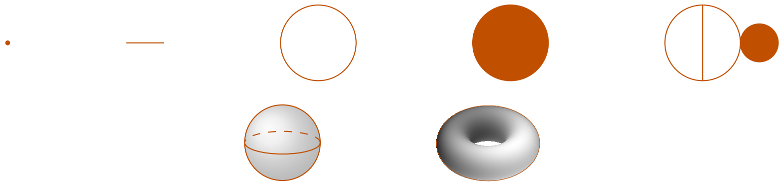

considered up to "deformation".

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In particular, given a point $x \in A$, there is always the **constant path** from x to x, defined by $f(i) = x$ for $i \in I$.

Identity types in spaces

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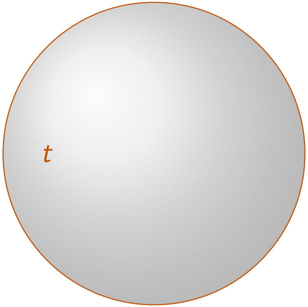

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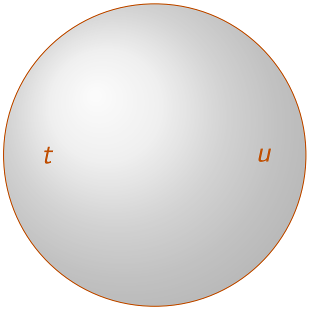

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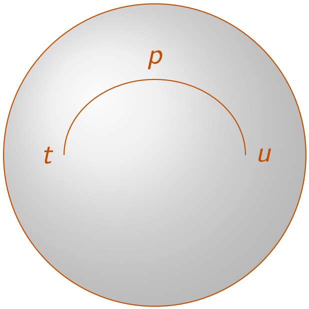

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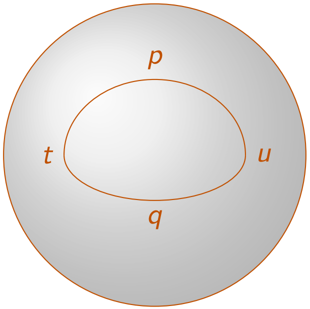

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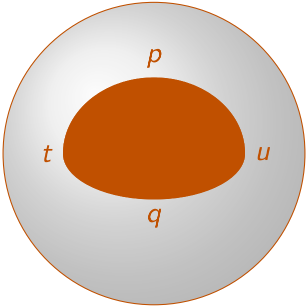

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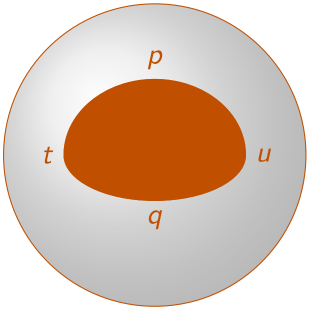
- and so on.



Homotopy equivalence

Two functions $f : A \to B$ and $g : B \to A$ between spaces are **homotopic** when

- $f(x)$ can be deformed into $g(x)$, i.e. there is a path from $f(x)$ to $g(x)$,
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Two spaces A and B are **homotopy equivalent** when there is

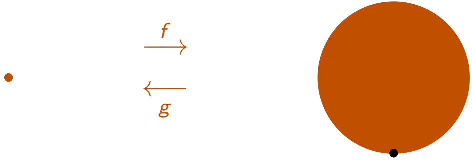
$$f : A \to B \qquad \text{and} \qquad g : B \to A$$

such that

$$g \circ f \sim \mathrm{id}_A \qquad \text{and} \qquad f \circ g \sim \mathrm{id}_B$$

Homotopy equivalence

For instance, the following spaces are homotopy equivalent:



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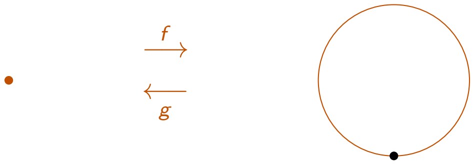

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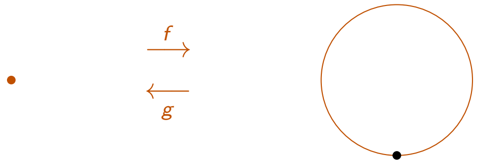

It can be shown that equivalent spaces always have the same number of "holes" in every dimension (and this can even be taken as a definition).

Homotopy type theory

Because it considers spaces up to homotopy equivalence, the resulting theory is called **homotopy type theory**.

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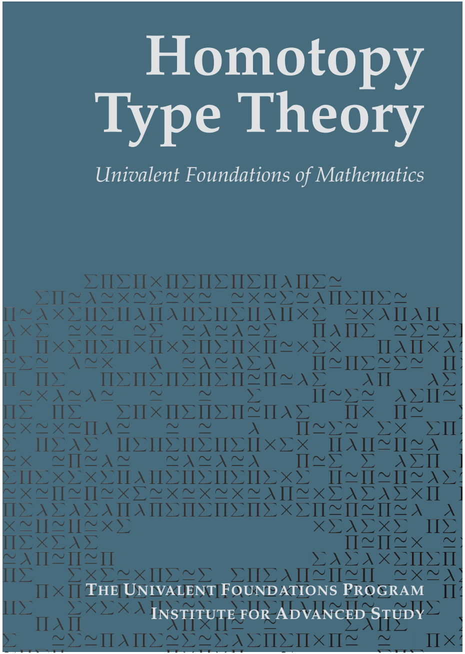
In order to make this clear, we write `Type` instead of `Set` in the following.


Homotopy
Type Theory
Univalent Foundations of Mathematics
The Univalent Foundations Program
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The elimination principle says that in order to show property depending on a path p, it is enough to show it for the constant path `refl`.

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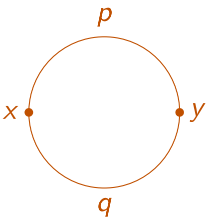


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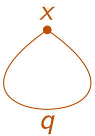


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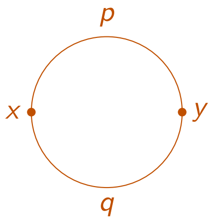


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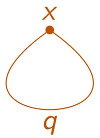


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If we consider spaces up to homotopy we should be careful!

```
I'm not sure if there should be a case for the constructor refl,
because I get stuck when trying to solve the following unification
problems (inferred index =? expected index):
  x₁ =? x₁
Possible reason why unification failed:
  Cannot eliminate reflexive equation x₁ = x₁ of type A₁ because K
  has been disabled.
when checking that the expression ? has type refl ≡ q
```

The structure of paths

Three important constructions on paths:

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  ! : {x y : A} → x ≡ y → y ≡ x
  ! refl = refl
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•-assoc : {x y z w : A} →
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```
•-inv-l : {x y : A} → (p : x ≡ y) → ! p • p ≡ refl
•-inv-l refl = refl
•-inv-r : {x y : A} → (p : x ≡ y) → p • ! p ≡ refl
•-inv-r refl = refl
```

The structure of paths

This structure is like a group, excepting that we can only compose paths when their target and source endpoints match: this is called a **groupoid**.

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Moreover, note that the laws are not exactly satisfied: they are only so up to higher paths...

Part IV

n-types

Classifying types

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$$\text{TYPE} = \text{SPACE}$$

we can begin to think of a classification of types.

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$$\text{equality proof} = \text{path}$$

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But the booleans are not a proposition:

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Bool-isnt-prop : ¬ (isProp Bool)
Bool-isnt-prop P with P true false
Bool-isnt-prop P | ()
```

Propositions

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(we are really ignoring universes here)

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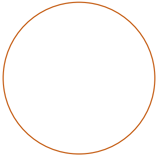

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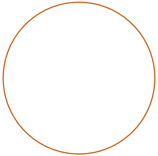

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but it is not so, because all the functions we can write are continuous!

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Similarly, for any set A, the set $A \to 0$ of functions (= implications) contains either 1 or 0 elements (depending on whether A is empty or not). We thus expect $\neg A$ to be a proposition for any A.

Propositions

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isProp-∧ : {A B : Type} → isProp A → isProp B → isProp (A × B)
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because we do not have any useful tool to show the equality of functions.

Function extensionality

In fact, we need **function extensionality**:

```
postulate funext : {A : Type} {B : A → Type} → {f g : (x : A) → B x} →
                   ((x : A) → f x ≡ g x) → f ≡ g
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We already mentioned that this axiom was not reasonable, because we want to capture intensional properties of functions.

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However, in a homotopic setting, this does not say that `f` is the same as `g`, only that one can be deformed to the other.

Moreover, we will see that it actually follows from the main (only?) axiom of homotopy type theory: *univalence*.

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isProp-¬ : {A : Type} → isProp (¬ A)
isProp-¬ ¬x ¬y = funext (λ x → ⊥-elim (¬x x))
```

Propositions

The fact of being a proposition is itself a proposition:

```
isProp-isProp : {A : Type} → isProp (isProp A)
```

(set later on for the proof).

Propositions

Note : we started Curry-Howard as

$$\text{propositions} = \text{types}$$

but what we really have is

$$\text{propositions} \subseteq \text{types}$$

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The next thing we can define are **sets**.

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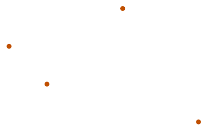
We should follow the idea that a set consists of points:



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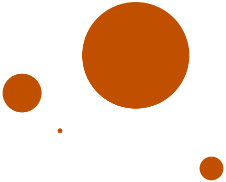

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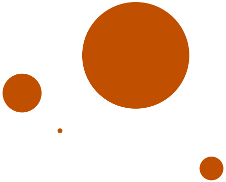

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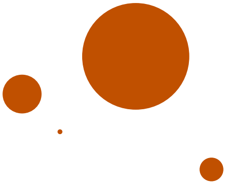

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as well as natural numbers:

```
suc-≡ : {m n : ℕ} → (p : suc m ≡ suc n) →
        Σ (m ≡ n) (λ q → cong suc q ≡ p)
suc-≡ refl = refl , refl

ℕ-isSet : isSet ℕ
ℕ-isSet zero .zero refl refl = refl
ℕ-isSet (suc x) .(suc x) refl p with (suc-≡ p)
... | q , e = trans (cong (cong suc) (ℕ-isSet x x refl q)) e
```

More generally,

Theorem (Hedberg)
Any type with a decidable equality is a set.

Sets

Also, propositions are sets:

```
aProp-isSet : {A : Type} → isProp A → isSet A
```

We can notice a "pattern" (of length one...): a set is a type in which there is a at most one equality (up to homotopy) between two elements:

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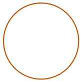
```
is1Type : Type → Type
is1Type A = (x y : A) → isSet (x ≡ y)
```

1-types

A 1-type is type in which there is at most one equality between two equalities:

- the circle is a 1-type:



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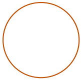

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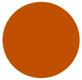

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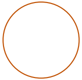

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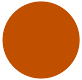

- the sphere is not a 1-type:

n-types

From now on, the definition of n-types is clear:

- a 0-type is a set (by convention),
- an $(n+1)$-type is a type in which $\mathtt{x} \equiv \mathtt{y}$ is an n-type for every elements $\mathtt{x}$ and $\mathtt{y}$.

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(-2)-types

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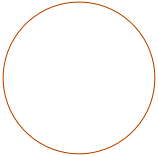

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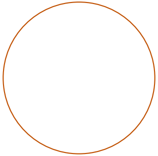

is a (-2)-type

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isContr : Type → Type
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```

but it is not so because functions have to be continuous.

n-types

In Agda, we can thus define (starting at 0 instead of -2):

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```
hasLevel : ℕ → Type → Type
hasLevel  zero    A = isContr A
hasLevel (suc n) A = (x y : A) → hasLevel n (x ≡ y)
```

n-types

We can show interesting properties such as:

Theorem
Any n-type is an $(n+1)$-type.

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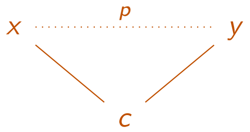
Theorem
Any n-type is an $(n+1)$-type.

```
cumulative : (n : ℕ) {A : Type} → hasLevel n A → hasLevel (suc n) A
cumulative zero L x y =
  (! (snd L x) • snd L y) , λ { refl → •-inv-l (snd L x) }
cumulative (suc n) L x y = cumulative n (L x y)
```



n-types

Or that

Theorem
Being an n-type is a proposition.

Or that

Theorem
Being an n-type is a proposition.

For instance,

```
isProp-isProp : {A : Type} → isProp (isProp A)
isProp-isProp {A = A} f g =
  funext2 {f = f} {g = g} (λ x y → aProp-isSet g x y (f x y) (g x y))
```

where `funext2` is function extensionality for functions with two arguments.

Part V

Univalence

Let's see some other operations available with paths.

Application

Functions respect identities (intuitively, because they are continuous):

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```
ap : {A B : Type} {x y : A} → (f : A → B) → x ≡ y → f x ≡ f y
ap f refl = refl
```

In other words, we can **apply** a function to a path.

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In other words, we can **apply** a function to a path.

This is what we called `cong` before.

Lemma
Application is compatible with concatenation.

Application

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Application is compatible with concatenation.

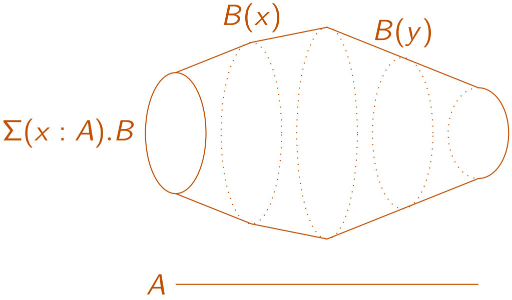
Proof.

```
•-ap : {A B : Type} {x y z : A} → (f : A → B) →
       (p : x ≡ y) → (q : y ≡ z) → ap f (p • q) ≡ ap f p • ap f q
•-ap f refl q = refl
```

□

Type families

A type family `P : A → Type` should be thought of as a collection of spaces `P a` for each `a : A` which varies *continuously* in `a`:



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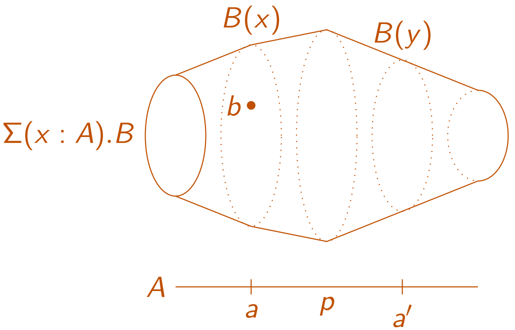

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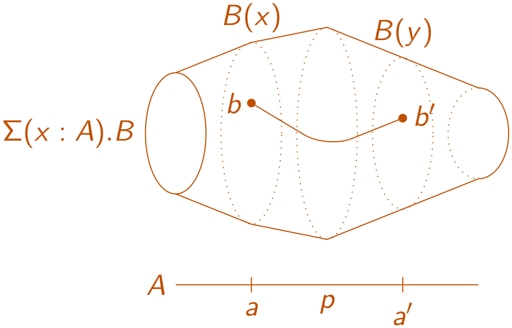

Given a path p in A from a to a' and a point b in $P\,a$ we expect that there is a unique path in P whose "projection" on A is p.

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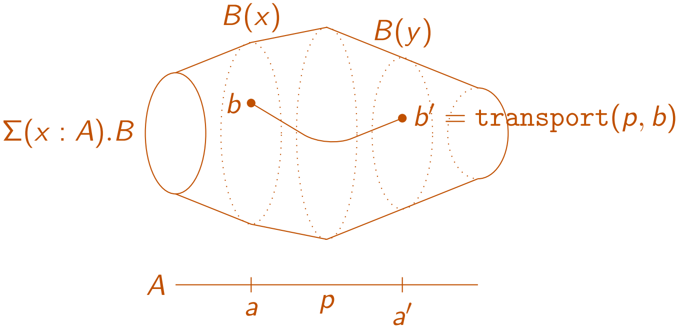

Given a path p in A from a to a' and a point b in $P\ a$ we expect that there is a unique path in P whose "projection" on A is p.

We call its other end in $P\ b$, the **transport** of b along p.

Transport

Formally, the **transport** operation is defined as

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transport : {A : Type} {x y : A} (P : A → Type) → x ≡ y → P x → P y
transport P refl x = x
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transport : {A : Type} {x y : A} (P : A → Type) → x ≡ y → P x → P y
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This is what we called `subst` before.

Transport

We can show that transporting a path along one of its end amounts to composing it with the path:

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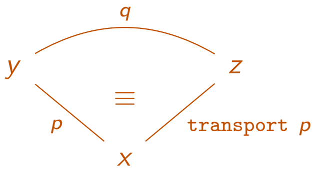
```
transport-≡-r : {A : Type} {x y z : A} → (p : x ≡ y) (q : y ≡ z) →
                transport (λ y → x ≡ y) q p ≡ (p • q)
transport-≡-r p refl = sym (•-unit-r p)
```



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```

and similarly on the other side:

```
transport-≡-l : {A : Type} {x y z : A} → (p : x ≡ y) (q : y ≡ z) →
                transport (λ y → y ≡ z) (! p) q ≡ (p • q)
transport-≡-l refl q = refl
```

Dependent application

We have defined application of a function $f : A \to B$ to a path $p : x \equiv y$ in A:

```
ap : {A B : Type} {x y : A} → (f : A → B) → x ≡ y → f x ≡ f y
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We are thus tempted to prove

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apd : {A : Type} {B : A → Type} {x y : A} →
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What is the problem?

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What is the problem?

The correct definition of **dependent application** is

```
apd : {A : Type} {B : A → Type} {x y : A} → (f : (x : A) → B x) →
      (p : x ≡ y) → transport B p (f x) ≡ f y
apd f refl = refl
```

Dependent application

Lemma
Every proposition A *is a set:*

Dependent application

Lemma
Every proposition A is a set:

```
aProp-isSet : {A : Type} → isProp A → isSet A
aProp-isSet {A} P x y p q = ?
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aProp-isSet : {A : Type} → isProp A → isSet A
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```

```
I'm not sure if there should be a case for the constructor refl,
because I get stuck when trying to solve the following unification
problems (inferred index =? expected index):
  x₁ =? x₁
Possible reason why unification failed:
  Cannot eliminate reflexive equation x₁ = x₁ of type A₁ because K
  has been disabled.
when checking that the expression ? has type refl ≡ q
```

Dependent application

Lemma
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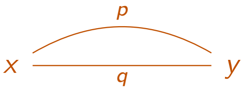

Proof.

Given two paths $p, q : x \equiv y$, we have to show $p \equiv q$.

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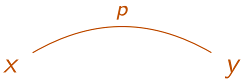

Proof.

Given two paths $p, q : x \equiv y$, we have to show $p \equiv q$. Consider p

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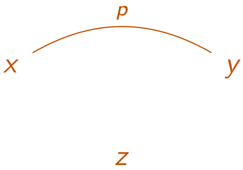

Proof.

Given two paths $p, q : x \equiv y$, we have to show $p \equiv q$. Consider p and take a point z.

Dependent application

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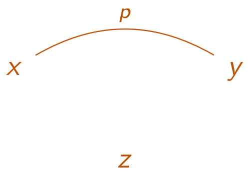

Proof.

Given two paths $p, q : x \equiv y$, we have to show $p \equiv q$. Consider p and take a point z.
Since A is a proposition, we have a function $f : (x : A) \to z \equiv x$.

Dependent application

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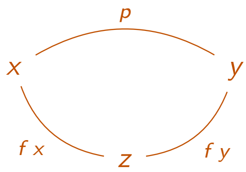

Proof.

Given two paths $p, q : x \equiv y$, we have to show $p \equiv q$. Consider p and take a point z. Since A is a proposition, we have a function $f : (x : A) \to z \equiv x$. In particular, we can consider $f\ x$ and $f\ y$.

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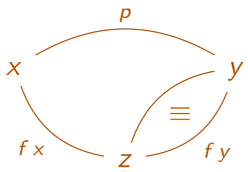

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Dependent application

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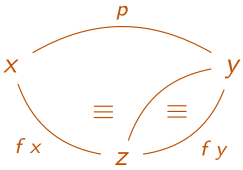

Proof.
Given two paths $p, q : x \equiv y$, we have to show $p \equiv q$. Consider p and take a point z. Since A is a proposition, we have a function $f : (x : A) \to z \equiv x$. In particular, we can consider $f\, x$ and $f\, y$. Using $\texttt{apd}$ of f to p we get a path from $\texttt{transport}\,(f\, x)\, p$ to $f\, y$ but the first is equal to $f\, x \cdot p$.

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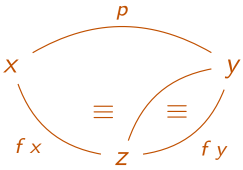
Every proposition A is a set:



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Dependent application

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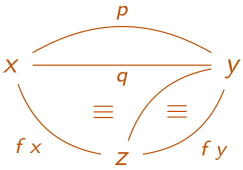

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Dependent application

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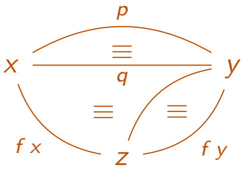

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Dependent application

Note that in order to show that $p \equiv (f\,x)^{-1} \cdot f\,y$, we could also have done an induction on p and shown the result in the case where p is `refl`, i.e.

$$\texttt{refl} \equiv (f\,x)^{-1} \cdot f\,x$$

which we have already shown.

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Formally,

```
aProp-isSet : {A : Type} → isProp A → isSet A
aProp-isSet {A} P x y p q = trans (lem x p) (sym (lem x q))
  where
  lem : (z : A) (p : x ≡ y) → p ≡ ! (P z x) • (P z y)
  lem z refl = sym (•-inv-l (P z x))
```

Homotopy

Two functions are **homotopic** when they are extensionally equal:

```
_∼_ : {A : Type} {B : A → Type} (f g : (x : A) → B x) → Type
_∼_ {A} f g = (x : A) → f x ≡ g x
```

This relation is different from equality between functions
(if we do not assume function extensionality or some other axiom).

Equivalences

We can define the identity function on a type A by

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id x = x
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id : {A : Type} → A → A
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We can define the composition of functions by

```
_∘_ : {A : Type} {B : Type} {C : Type} → (B → C) → (A → B) → (A → C)
(g ∘ f) x = g (f x)
```

Equivalences

A function is an **equivalence** when

Equivalences

A function is an **equivalence** when

```
isEquiv : {A : Type} {B : Type} → (A → B) → Type
isEquiv {A} {B} f =
  Σ (B → A) (λ g → (f ∘ g) ∼ id) × Σ (B → A) (λ g → (g ∘ f) ∼ id)
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A function is an **equivalence** when

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```
_≃_ : (A B : Type) → Type
A ≃ B = Σ (A → B) isEquiv
```

Equivalences

It seems that we could have defined equivalences more simply as

```
isEquiv' : {A : Type} {B : Type} → (A → B) → Type
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but this is not equivalent to the previous definition.

In fact this not the right definition, one way to see this is that we have

```
isEquiv-isProp : {A : Type} {B : Type} (f : A → B) → isProp (isEquiv f)
```

but there exists a function `f : A → B` such that this does not hold for `isEquiv’`.

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Note that we need to be serious about (cumulative) universes here since we have an equivalence between a small and a big type.

Univalence

The most useful consequence of this is that we have a map

```
ua : {A B : Type} → (A ≃ B) → (A ≡ B)
ua {A} {B} f with univalence A B
ua {A} {B} f | (g , _) , _ = g f
```

which allows us to make an equality from an equivalence, for which we have the usual tools such as transport.

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  b0 : Bin
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```
Bin-to-Nat : Bin → ℕ
Bin-to-Nat b0 = 0
Bin-to-Nat (b1 []) = 1
Bin-to-Nat (b1 (x ∷ l)) = (if x then 1 else 0) + 2 * Bin-to-Nat (b1 l)
```

Unary and binary natural numbers

This function can be shown to induce an equivalence between the two representations:

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Bin-to-Nat-isEquiv : isEquiv (Bin-to-Nat)
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We can therefore define addition on binary numbers from the one on unary numbers:

```
add : Bin → Bin → Bin
add = transport
        (λ A → (A → A → A))
        (sym (ua (Bin-to-Nat , Bin-to-Nat-isEquiv)))
        _+_
```

Univalence

In order to understand better univalence, it is simpler to take a variant of

```
id-to-equiv : {A B : Type} → (A ≡ B) → (A ≃ B)
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- **uniqueness** rule: for every $p : A \equiv B$,

$$\mathtt{ua}\,(\mathtt{coe}\,p) \equiv p$$

Univalence and classical logic

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- by transport and happly we can show that we should have $a \cong \mathtt{not}\, a$,
- we therefore have $\mathtt{true} \equiv \mathtt{false}$ from which we can deduce $\bot$.

Univalence and classical logic

However, there is no contradiction in supposing

$$\neg\neg A \to A$$

for every proposition A.

Higher inductive types

For now, we don't have many non-trivial 2-types at our disposal (excepting SET).

Namely, all the types we constructed up to now are sets
(natural numbers, lists over sets, etc.).

For instance, there is no easy way to construct something which looks like a circle.

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Namely, all the types we constructed up to now are sets
(natural numbers, lists over sets, etc.).

For instance, there is no easy way to construct something which looks like a circle.

In order to do so, we need a generalization of inductive types: **higher inductive types**.

Higher inductive types

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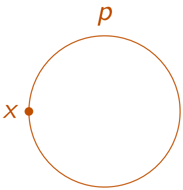
Those are not completely well understood (and implemented) as of now.

Higher inductive types

For instance, we can define the circle



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data Circle : Type where
  x : Circle
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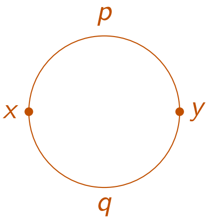

as

```
data Circle : Type where
  x : Circle
  y : Circle
  p : x ≡ y
  q : x ≡ y
```

Higher inductive types

Recall that for booleans

```
data Bool : Type where
  true  : Bool
  false : Bool
```

the recursion principle is that given

- a type A,
- an element $t : A$
- an element $u : A$

there exists a unique function

$$f : \mathrm{Bool} \to A$$

such that $f\,\texttt{true} = t$ and $f\,\texttt{false} = u$.

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such that $f\,\texttt{true} = t$ and $f\,\texttt{false} = u$.
Exercise: define `not`.

Higher inductive types

If we consider the type

```
data Circle : Type where
  base : Circle
  loop : base ≡ base
```

the corresponding induction principle is that given

- a predicate $P : \mathtt{Circle} \to \mathsf{Type}$,
- an element $b : P\,\mathtt{base}$,
- a path $l : P\,\mathtt{base} \equiv P\,\mathtt{base}$

there exists a (unique up to homotopy) function

$$f : (x : \mathtt{Circle}) \to P\,x$$

such that $f\,\mathtt{base} = b$ and $f\,\mathtt{loop} = l$.

Suspension

The **suspension** of a type *A* is

```
data Susp (A : Type) : Type where
  N : Susp A
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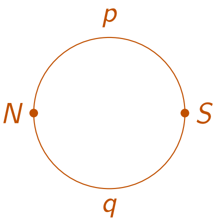

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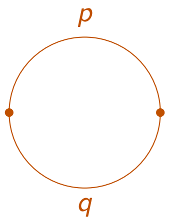

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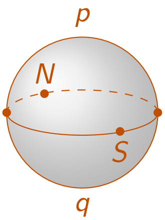

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data Susp (A : Type) : Type where
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In this way, we can construct the *n*-sphere for any dimension *n*...

Propositional truncation

The **propositional truncation** of a type is

```
data Trunc (A : Type) : Type where
  carrier : A → Trunc A
  trivial : (x y : Trunc A) → x ≡ y
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It can be thought of as turning a type A into a proposition,
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It can be thought of as turning a type A into a proposition,
i.e. it is (equivalent to) a point when A is non-empty and a empty when A is empty.

However, this is done in an intuitionistic way (the above would rather be $\neg\neg A$).

Propositional truncation

The recursion principle says that given

- a type B,
- a function $g : A \to B$,
- a path $x \equiv y$ for every $x, y : B$,

there exists a unique function

$$f : \mathtt{Trunc}\, A \to B$$

such that $f\,x = g\,x$ for $x : A$ and given $x, y : A$, $\mathrm{ap}\, f$ sends the specified path from x to y in A to the one between $f\,x$ and $f\,y$ in B.

Propositional truncation

For instance, there is a canonical map from $\texttt{Trunc}\, A$ to $\neg\neg A$ induced by

- the map $A \to \neg\neg A$ sending x to $\lambda f. f\, x$,
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It is an equivalence if and only if the logic is classical.