CSC_51051_EP: Natural deduction

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Part I

Introduction

Propositional logic

In propositional logic, we consider **formulas** which are built from

- variables: X, Y, $\ldots$
- connectives: $\wedge$, $\vee$, $\Rightarrow$, $\neg$, $\ldots$

For instance,

$$(X \Rightarrow Y) \Rightarrow (\neg X \vee Y)$$

The boolean interpretation

Usually,

- we interpret variables as booleans (*valuations*),
- we have a standard interpretation for connectives

$A \wedge B$	0	1
0	0	0
1	0	1

$A \vee B$	0	1
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$A \Rightarrow B$	0	1
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A formula is **valid** when its interpretation is true for every value given to the variables.

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$A \land B$	0	1
0	0	0
1	0	1

$A \lor B$	0	1
0	0	1
1	1	1

$A \Rightarrow B$	0	1
0	1	1
1	0	1

$\neg A$	
0	1
1	0

A formula is **valid** when its interpretation is true for every value given to the variables.

$$\underbrace{(X \Rightarrow Y) \Rightarrow (\neg X \lor Y)}_{1}$$

with $X = 1$ and $Y = 0$.

The set-theoretic interpretation

With this idea that propositions should correspond to types and consider

$$\mathbb{N} \Rightarrow \mathbb{N}$$

We should therefore interpret

- a type A as a **set**,
- $A \Rightarrow B$ as
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- $A \lor B$ as

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- $A \wedge B$ as the product $A \times B$,
- $A \vee B$ as the disjoint union $A \sqcup B$.

We recover the previous interpretation by considering whether a set is empty or not:

$A \wedge B$	$\emptyset$	1
$\emptyset$	$\emptyset$	$\emptyset$
1	$\emptyset$	1

$A \vee B$	$\emptyset$	1
$\emptyset$	$\emptyset$	1
1	1	1

$A \Rightarrow B$	$\emptyset$	1
$\emptyset$	1	1
1	$\emptyset$	1

$\neg A$	
$\emptyset$	1
1	$\emptyset$

The proof-theoretic interpretation

This is not entirely satisfactory since the way a function is implemented matters.

For instance,

$$\begin{aligned} \mathbb{N} &\Rightarrow \mathbb{N} \\ n &\mapsto 0 \end{aligned}$$

can be implemented as

- `let f n = 0`

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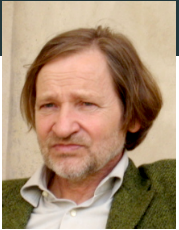
Note that the complexities are respectively $O(1)$, $O(\log_2(n))$ and $O(n)$.

The three levels of interpretation



Girard advocates that there are three levels for interpreting proofs:

0. the *boolean level*: propositions are interpreted as booleans and we are interested in whether a proposition is provable or not,
1. the *extensional level*: propositions are interpreted as sets and we are interested in which functions can be implemented,
2. the *intentional level*: we are interested in the proofs (= programs) themselves, and how they evolve during reduction,

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2. the *intentional level*: we are interested in the proofs (= programs) themselves, and how they evolve during reduction,

$\vdots$

∞. the *homotopical level*: we take equality seriously in account.

Intuitionism

Shifting from provability to proofs was initiated by Brouwer's **intuitionism** (around 1900):

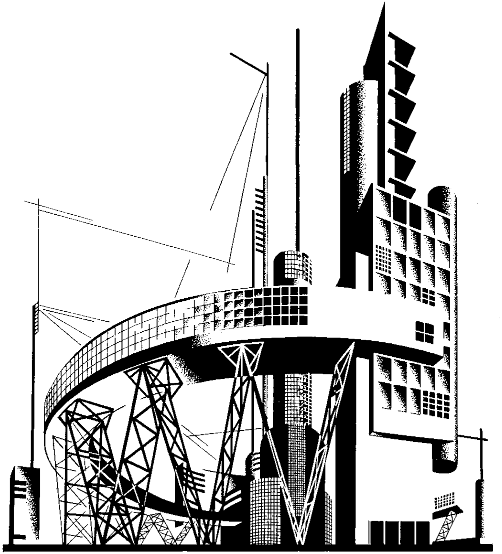
- mathematics is not about a preexisting reality,
  it is a subjective mental construction,
- this construction has an existence of its own.

In this point of view, we switch from provability to proofs:

- $A \wedge B$ means that I have both a proof of A and a proof B
  (it is a product rather than an intersection)
- $A \Rightarrow B$ is a way of producing a proof of B from a proof of A.

This has some important consequences that we will see.

Constructivism



Formalizing the notion proof

We want to give a precise definition of the notion of **proof**.

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For instance, we want to show that $x \mapsto 2 \times x$ is continuous in 0:
we have to prove the formula

$$\forall \varepsilon . (\varepsilon > 0 \Rightarrow \exists \eta . (\eta > 0 \land \forall x . |x| < \eta \Rightarrow |2x| < \varepsilon))$$

Formalizing the notion proof

Goal:

$$\forall \varepsilon.(\varepsilon > 0 \Rightarrow \exists \eta.(\eta > 0 \wedge \forall x.|x| < \eta \Rightarrow |2x| < \varepsilon))$$

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- Suppose given ε.

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Goal:

$$\exists \eta.(\eta > 0 \land \forall x. |x| < \eta \Rightarrow |2x| < \varepsilon)$$

- Suppose given ε.
- Suppose $\varepsilon > 0$.

Formalizing the notion proof

Goal:

$$\varepsilon/2 > 0 \wedge \forall x. |x| < \varepsilon/2 \Rightarrow |2x| < \varepsilon$$

- Suppose given ε.
- Suppose $\varepsilon > 0$.
- Take $\eta = \varepsilon/2$.

Formalizing the notion proof

Goals:

$$\varepsilon/2 > 0 \qquad \text{and} \qquad \forall x. |x| < \varepsilon/2 \Rightarrow |2x| < \varepsilon$$

- Suppose given ε.
- Suppose $\varepsilon > 0$.
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Formalizing the notion proof

Goals:

$$2 \times (\varepsilon/2) > 2 \times 0 \qquad \text{and} \qquad \forall x. |x| < \varepsilon/2 \Rightarrow |2x| < \varepsilon$$

- Suppose given ε.
- Suppose $\varepsilon > 0$.
- Take $\eta = \varepsilon/2$.
- Since $2 > 0$.

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- Suppose given ε.
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- Since $2 > 0$.
- By usual identities.

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Goal:

$$\forall x. |x| < \varepsilon/2 \Rightarrow |2x| < \varepsilon$$

- Suppose given ε.
- Suppose $\varepsilon > 0$.
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- Since $2 > 0$.
- By usual identities.
- By hypothesis.

Formalizing the notion proof

Goal:

$$|x| < \varepsilon/2 \Rightarrow |2x| < \varepsilon$$

- Suppose given ε.
- Suppose $\varepsilon > 0$.
- Take $\eta = \varepsilon/2$.

- Since $2 > 0$.
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- By hypothesis.

- Suppose given x.

Formalizing the notion proof

Goal:

$$|2x| < \varepsilon$$

- Suppose given ε.
- Suppose $\varepsilon > 0$.
- Take $\eta = \varepsilon/2$.

- Since $2 > 0$.
- By usual identities.
- By hypothesis.

- Suppose given x.
- Suppose $|x| < \varepsilon/2$.

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Goal:

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- Suppose given ε.
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Formalizing the notion proof

Done!

- Suppose given ε.
- Suppose $\varepsilon > 0$.
- Take $\eta = \varepsilon/2$.

- Since $2 > 0$.
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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varepsilon > 0 \vdash \varepsilon > 0}}{\varepsilon > 0 \vdash (\varepsilon/2) \times 2 > 0 \times 2}}{\varepsilon > 0 \vdash \varepsilon/2 > 0} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varepsilon > 0, |x| < \varepsilon/2 \vdash |x| < \varepsilon/2}}{\varepsilon > 0, |x| < \varepsilon/2 \vdash |2x|/2 < \varepsilon/2}}{\varepsilon > 0, |x| < \varepsilon/2 \vdash |2x| < \varepsilon}}{\varepsilon > 0 \vdash |x| < \varepsilon/2 \Rightarrow |2x| < \varepsilon}}{\varepsilon > 0 \vdash \forall x.|x| < \varepsilon/2 \Rightarrow |2x| < \varepsilon}}{\varepsilon > 0 \vdash \varepsilon/2 > 0 \wedge \forall x.|x| < \varepsilon/2 \Rightarrow |2x| < \varepsilon}}{\varepsilon > 0 \vdash \exists \eta.(\eta > 0 \wedge \forall x.|x| < \eta \Rightarrow |2x| < \varepsilon)}}{\vdash \varepsilon > 0 \Rightarrow \exists \eta.(\eta > 0 \wedge \forall x.|x| < \eta \Rightarrow |2x| < \varepsilon)}}{\vdash \forall \varepsilon.(\varepsilon > 0 \Rightarrow \exists \eta.(\eta > 0 \wedge \forall x.|x| < \eta \Rightarrow |2x| < \varepsilon))}$$

Formalizing the notion of proof

This has several interesting consequences.

- We can show properties of the system: consistency, decidability, etc.
- We can reason on proofs.
- We can transform proofs.

Part II

Intuitionistic natural deduction

Formulas

We consider **formulas** generated by the grammar

$$A, B ::= X \mid A \Rightarrow B \mid A \wedge B \mid \top \mid A \vee B \mid \bot \mid \neg A$$

where X is a variable.

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- the operations are bracketed on the *right*:

$$A_1 \wedge A_2 \wedge A_3 \Rightarrow B \Rightarrow C \qquad \text{is} \qquad (A_1 \wedge (A_2 \wedge A_3)) \Rightarrow (B \Rightarrow C)$$

Sequents

A **context** Γ is a list of formulas

$$A_1, \ldots, A_n$$

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An **inference rule**

$$\frac{\Gamma_1 \vdash A_1 \qquad \ldots \qquad \Gamma_n \vdash A_n}{\Gamma \vdash A}$$

specifies when we can deduce a sequent from others.

Intuitionistic natural deduction (NJ)

$$\frac{}{\Gamma, A, \Gamma' \vdash A}(\mathrm{ax})$$

$$\frac{\Gamma \vdash A \Rightarrow B \qquad \Gamma \vdash A}{\Gamma \vdash B}(\Rightarrow_{\mathrm{E}})$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}(\Rightarrow_{\mathrm{I}})$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}(\wedge_{\mathrm{E}}^{\mathrm{l}}) \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}(\wedge_{\mathrm{E}}^{\mathrm{r}})$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}(\wedge_{\mathrm{I}})$$

$$\frac{\Gamma \vdash A \vee B \qquad \Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma \vdash C}(\vee_{\mathrm{E}})$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}(\vee_{\mathrm{I}}^{\mathrm{l}}) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}(\vee_{\mathrm{I}}^{\mathrm{r}})$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A}(\bot_{\mathrm{E}})$$

$$\frac{}{\Gamma \vdash \top}(\top_{\mathrm{I}})$$

$$\frac{\Gamma \vdash \neg A \qquad \Gamma \vdash A}{\Gamma \vdash \bot}(\neg_{\mathrm{E}})$$

$$\frac{\Gamma, A \vdash \bot}{\Gamma \vdash \neg A}(\neg_{\mathrm{I}})$$

Rules: remarks

Apart from the axiom, rules are either

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Some rules have latin names:

- $(\Rightarrow_{\mathsf{E}})$: *modus ponens*
- $(\bot_{\mathsf{E}})$: *ex falso quot libet* or *explosion principle*

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A formula A is **provable** when the sequent $\vdash A$ is.

A proof of $A \Rightarrow A$

$$\vdash A \Rightarrow A$$

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$$\frac{A \vdash A}{\vdash A \Rightarrow A}\,(\Rightarrow_{\mathrm{I}})$$

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$$\dfrac{\dfrac{}{A \vdash A}\,(\mathrm{ax})}{\vdash A \Rightarrow A}\,(\Rightarrow_{\mathrm{I}})$$

A proof of $A \wedge B \Rightarrow A \vee B$

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A proof of $A \land B \Rightarrow A \lor B$

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A proof of $(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$

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$$\frac{A \Rightarrow B \vdash \neg B \Rightarrow \neg A}{\vdash (A \Rightarrow B) \Rightarrow \neg B \Rightarrow \neg A}\;(\Rightarrow_{\mathsf{I}})$$

A proof of $(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$

$$\dfrac{\dfrac{A \Rightarrow B, \neg B \vdash \neg A}{A \Rightarrow B \vdash \neg B \Rightarrow \neg A}\,(\Rightarrow_{\mathrm{I}})}{\vdash (A \Rightarrow B) \Rightarrow \neg B \Rightarrow \neg A}\,(\Rightarrow_{\mathrm{I}})$$

A proof of $(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$

$$\dfrac{\dfrac{\dfrac{A \Rightarrow B, \neg B, A \vdash \bot}{A \Rightarrow B, \neg B \vdash \neg A}\,(\neg_{\mathrm{I}})}{A \Rightarrow B \vdash \neg B \Rightarrow \neg A}\,(\Rightarrow_{\mathrm{I}})}{\vdash (A \Rightarrow B) \Rightarrow \neg B \Rightarrow \neg A}\,(\Rightarrow_{\mathrm{I}})$$

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$$
\dfrac{
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\dfrac{}{A \Rightarrow B, \neg B, A \vdash \neg B}\,(\mathrm{ax})
\qquad
\dfrac{A \Rightarrow B, \neg B, A \vdash A \Rightarrow B \qquad A \Rightarrow B, \neg B, A \vdash A}{A \Rightarrow B, \neg B, A \vdash B}\,(\Rightarrow_{\mathrm{E}})
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Definable connectives

We could have added some other connectives such as $\Leftrightarrow$, which can be implemented:

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Even some present connectives can be implemented from others:

$$\neg A \qquad = \qquad A \Rightarrow \bot$$

Proof.
Can be deduced (not trivial) from the fact that $\neg A \Leftrightarrow (A \Rightarrow \bot)$ is provable. $\square$

Part III

Properties of NJ

Induction on proofs

Since proofs have become syntactic objects,
we can reason (= make proofs) on them!

Induction on natural numbers

The set $\mathbb{N}$ of natural numbers is the smallest set containing 0 and closed under successor.

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The **recurrence principle** states that, given a predicate $P(n)$ on natural numbers, if

- $P(0)$ and
- for $n \in \mathbb{N}$, $P(n)$ implies $P(S(n))$,

then $P(n)$ holds for every $n \in \mathbb{N}$.

Induction on natural numbers

The set $\mathbb{N}$ of natural numbers is the smallest set containing 0 and closed under successor.

$$\frac{}{0 : \mathbb{N}} \qquad\qquad \frac{n : \mathbb{N}}{S(n) : \mathbb{N}}$$

The **recurrence principle** states that, given a predicate $P(n)$ on natural numbers, if, for every rule,

- if P holds for the premises then P holds for the conclusion

then $P(n)$ holds for every $n \in \mathbb{N}$.

Induction on proofs

The set of proofs is the smallest set closed under the deduction rules:

Theorem

Suppose given a predicate $P(\pi)$ on proofs π. Suppose that, for every rule,

$$\pi \quad = \quad \frac{\dfrac{\pi_1}{\Gamma_1 \vdash A_1} \qquad \ldots \qquad \dfrac{\pi_n}{\Gamma_n \vdash A_n}}{\Gamma \vdash A}$$

if $P(\pi_i)$ holds for every i then $P(\pi)$ also holds. Then $P(\pi)$ holds for every proof π.

Admissible rules

A rule

$$\frac{\Gamma_1 \vdash A_1 \qquad \dots \qquad \Gamma_n \vdash A_n}{\Gamma \vdash A}$$

is **admissible** when, whenever all the premises are provable, the conclusion is also provable.

This means that, from a proof for each $\Gamma_i \vdash A_i$, we can construct a proof of $\Gamma \vdash A$ (but not necessarily that we can implement the above deduction with some rules).

Weakening

We have this idea that if we can prove something with some hypothesis, then we can also prove it with more hypothesis.

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For instance:

$$\dfrac{\dfrac{\dfrac{}{A \wedge B \vdash A \wedge B}\ (\mathsf{ax})}{A \wedge B \vdash A}\ (\wedge_{\mathsf{E}}^{\mathsf{l}})}{A \wedge B \vdash A \vee B}\ (\vee_{\mathsf{I}}^{\mathsf{l}})$$

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For instance:

$$\dfrac{\dfrac{\dfrac{}{C, A \wedge B \vdash A \wedge B}\,(\mathrm{ax})}{C, A \wedge B \vdash A}\,(\wedge^{\mathrm{l}}_{\mathrm{E}})}{C, A \wedge B \vdash A \vee B}\,(\vee^{\mathrm{l}}_{\mathrm{I}})$$

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This can be formalized by showing that the following **weakening** rule is admissible.

Weakening

We can always add more hypothesis in a context:

Proposition

*The **weakening rule** is admissible:* $\dfrac{\Gamma, \Gamma' \vdash B}{\Gamma, A, \Gamma' \vdash B}\ (\mathsf{wk})$.

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By induction on the proof of $\Gamma, \Gamma' \vdash B$.

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By induction on the proof of $\Gamma, \Gamma' \vdash B$.

- If the proof is of the form

  $$\dfrac{}{\Gamma, \Gamma' \vdash B}\ (\mathrm{ax})$$

  with B occurring in Γ or Γ' then we conclude with

  $$\dfrac{}{\Gamma, A, \Gamma' \vdash B}\ (\mathrm{ax})$$

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*The **weakening rule** is admissible:* $\dfrac{\Gamma, \Gamma' \vdash B}{\Gamma, A, \Gamma' \vdash B}\ (\text{wk})$.

Proof.
By induction on the proof of $\Gamma, \Gamma' \vdash B$.

- If the proof is of the form $$\dfrac{\dfrac{\pi_1}{\Gamma, \Gamma' \vdash B \Rightarrow C} \qquad \dfrac{\pi_2}{\Gamma, \Gamma' \vdash B}}{\Gamma, \Gamma' \vdash C}\ (\Rightarrow_{\mathrm{E}})$$

  then we conclude with $$\dfrac{\dfrac{\pi_1'}{\Gamma, A, \Gamma' \vdash B \Rightarrow C} \qquad \dfrac{\pi_2'}{\Gamma, A, \Gamma' \vdash B}}{\Gamma, A, \Gamma' \vdash C}\ (\Rightarrow_{\mathrm{E}})\ .$$

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By induction on the proof of $\Gamma, \Gamma' \vdash B$.

- If the proof is of the form

  $$\dfrac{\dfrac{\pi}{\Gamma, \Gamma', B \vdash C}}{\Gamma, \Gamma' \vdash B \Rightarrow C}\ (\Rightarrow_{\text{I}})$$

  then we conclude with

  $$\dfrac{\dfrac{\pi'}{\Gamma, A, \Gamma', B \vdash C}}{\Gamma, A, \Gamma' \vdash B \Rightarrow C}\ (\Rightarrow_{\text{I}})$$

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*The **weakening rule** is admissible:* $\dfrac{\Gamma, \Gamma' \vdash B}{\Gamma, A, \Gamma' \vdash B}\,(\mathrm{wk})$.

Proof.
By induction on the proof of $\Gamma, \Gamma' \vdash B$.

- Other cases are similar. $\square$

Weakening

Note that the axiom rule

$$\frac{}{\Gamma, A, \Gamma' \vdash A}\ (\mathrm{ax})$$

is the only one really using the context, the other rules do not change the context, e.g.

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ (\wedge_{\mathrm{I}})$$

excepting

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\ (\Rightarrow_{\mathrm{I}})$$

which only adds one formula to the context.

This explains why most inductive cases go through without any difficulty, axiom being the only "subtle" one.

Contraction

In a context, the multiplicity of a formula does not really matter:

Proposition
The ***contraction rule*** *is admissible:*

$$\frac{\Gamma, A, A, \Gamma' \vdash B}{\Gamma, A, \Gamma' \vdash B}\ (\text{contr})$$

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Proof.
By induction on the proof of $\Gamma, A, A, \Gamma' \vdash C$:

$$\frac{}{\Gamma, A, A, \Gamma' \vdash B}\ (\text{ax}) \qquad \text{becomes}$$

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(including when $B = A$), other cases are simple, e.g.

$$\frac{\dfrac{\vdots}{\Gamma, A, \Gamma' \vdash B} \qquad \dfrac{\vdots}{\Gamma, A, \Gamma' \vdash C}}{\Gamma, A, \Gamma' \vdash B \wedge C}\ (\wedge_{\text{I}}) \qquad \text{becomes} \qquad \frac{\dfrac{\vdots}{\Gamma, A, A, \Gamma' \vdash B} \qquad \dfrac{\vdots}{\Gamma, A, A, \Gamma' \vdash C}}{\Gamma, A, A, \Gamma' \vdash B \wedge C}\ (\wedge_{\text{I}})$$

Exchange

In a context, the order of formulas does not really matter:

Proposition
The ***exchange rule***

$$\frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, B, A, \Gamma' \vdash C}\ (\text{xch})$$

is admissible.

Proof.
By induction on the proof of $\Gamma, A, B, \Gamma' \vdash C$.

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$$\frac{}{\Gamma, A, B, \Gamma' \vdash C}\ (\text{ax}) \qquad \text{becomes} \qquad \frac{}{\Gamma, B, A, \Gamma' \vdash C}\ (\text{ax})$$

other cases are immediate by induction. □

Truth strengthening

In a context, the formula $\top$ does not bring any information:

Proposition
The ***truth strengthening rule***

$$\frac{\Gamma, \top, \Gamma' \vdash A}{\Gamma, \Gamma' \vdash A}$$

is admissible.

Proof.
By induction on the proof of $\Gamma, \top, \Gamma' \vdash A$:

$$\frac{}{\Gamma, \top, \Gamma' \vdash \top}\ (\text{ax}) \qquad \text{becomes}$$

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is admissible.

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$$\frac{}{\Gamma, \top, \Gamma' \vdash \top}\ (\text{ax}) \qquad \text{becomes} \qquad \frac{}{\Gamma, \Gamma \vdash \top}\ (\top_\text{I})$$

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other cases are immediate by induction. □

Contexts as sets

The **structural rules**

$$\frac{\Gamma, \Gamma' \vdash B}{\Gamma, A, \Gamma' \vdash B}\ (\mathrm{wk}) \qquad \frac{\Gamma, A, A, \Gamma' \vdash B}{\Gamma, A, \Gamma' \vdash B}\ (\mathrm{contr})$$

$$\frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, B, A, \Gamma' \vdash C}\ (\mathrm{xch}) \qquad \frac{\Gamma, \top, \Gamma' \vdash A}{\Gamma, \Gamma' \vdash A}$$

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$$\frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, B, A, \Gamma' \vdash C}\,(\mathrm{xch}) \qquad \frac{\Gamma, \top, \Gamma' \vdash A}{\Gamma, \Gamma' \vdash A}$$

say that that contexts are commutative and idempotent.

Can we implement them as sets?

Contexts as sets

Quizz: how many “pure” OCaml programs of type

```
'a -> 'a -> 'a
```

can you come up with?

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Essentially two:

```
let f x y = x
let f x y = y
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Contexts as sets

Quizz: how many “pure” OCaml programs of type

```
'a -> 'a -> 'a
```

can you come up with?

Essentially two:

```
let f x y = x
let f x y = y
```

I say “essentially” because there are many other which are “equivalent”:

```
let f'  x y = (fun z -> z) x
let f'' x y = (fun (z,t) -> z) (x, 5)
...
```

Contexts as sets

Having contexts as sets is not a good idea: the two proofs

$$\dfrac{\dfrac{\dfrac{}{A, A \vdash A}\,(\mathrm{ax})}{A \vdash A \Rightarrow A}\,(\Rightarrow_{\mathrm{I}})}{\vdash A \Rightarrow A \Rightarrow A}\,(\Rightarrow_{\mathrm{I}}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A, A \vdash A}\,(\mathrm{ax})}{A, A \vdash A}\,(\mathrm{xch})}{A \vdash A \Rightarrow A}\,(\Rightarrow_{\mathrm{I}})}{\vdash A \Rightarrow A \Rightarrow A}\,(\Rightarrow_{\mathrm{I}})$$

get identified into

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,(\mathrm{ax})}{A \vdash A \Rightarrow A}\,(\Rightarrow_{\mathrm{I}})}{\vdash A \Rightarrow A \Rightarrow A}\,(\Rightarrow_{\mathrm{I}})$$

whereas we expect to have two projections!

Substitution

A **substitution** σ is a function which assigns a formula to every variable.

We write $A[\sigma]$ for the formula A with variables replaced according to σ.

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We write $A[\sigma]$ for the formula A with variables replaced according to σ.

Proposition
For every substitution σ*, if the sequent*

$$A_1, \dots, A_n \vdash A$$

is provable, then the sequent

$$A_1[\sigma], \dots, A_n[\sigma] \vdash A[\sigma]$$

is also provable.

Proof.
By induction on the proof of $A_1, \dots, A_n \vdash A$. □

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is provable, then the sequent

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is also provable.

For instance:

$$\cfrac{\vdots}{\vdash X \Rightarrow X} \qquad \rightsquigarrow \qquad \cfrac{\vdots}{\vdash (\bot \Rightarrow A \wedge B) \Rightarrow (\bot \Rightarrow A \wedge B)}$$

Proof substitution

Another more subtle operation is **proof substitution**:

Proposition
Given provable sequents

$$\frac{\pi}{\Gamma, A, \Gamma' \vdash B} \qquad \textit{and} \qquad \frac{\pi'}{\Gamma, \Gamma' \vdash A}$$

we can construct a proof

$$\frac{\pi[\pi'/A]}{\Gamma, \Gamma' \vdash B}$$

Proof substitution

For instance, given the proofs

$$\pi = \dfrac{\dfrac{\dfrac{}{\Gamma, A, B \vdash A}\,(\text{ax}) \qquad \dfrac{}{\Gamma, A, B \vdash A}\,(\text{ax})}{\Gamma, A, B \vdash A \wedge A}\,(\wedge_\text{I})}{\Gamma, A \vdash B \Rightarrow A \wedge A}\,(\Rightarrow_\text{I}) \qquad \text{and} \qquad \pi' = \dfrac{\vdots}{\Gamma \vdash A}$$

Proof substitution

For instance, given the proofs

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we can construct the following proof $\pi[\pi'/A]$:

$$\cfrac{\cfrac{\cfrac{\cfrac{\pi'}{\Gamma \vdash A}}{\Gamma, B \vdash A}\,(\text{wk}) \qquad \cfrac{\cfrac{\pi'}{\Gamma \vdash A}}{\Gamma, B \vdash A}\,(\text{wk})}{\Gamma, B \vdash A \wedge A}\,(\wedge_{\mathrm{I}})}{\Gamma \vdash B \Rightarrow A \wedge A}\,(\Rightarrow_{\mathrm{I}})$$

Proof substitution

Proposition

Given provable sequents $\dfrac{\pi}{\Gamma, A, \Gamma' \vdash B}$ *and* $\dfrac{\pi'}{\Gamma, \Gamma' \vdash A}$,

we can construct a proof $\dfrac{\pi[\pi'/A]}{\Gamma, \Gamma' \vdash B}$.

Proof.
By induction on π:

- we replace axioms $\dfrac{}{\Gamma, A, \Gamma', \Gamma'' \vdash A}$ (ax) by $\dfrac{w(\pi')}{\Gamma, \Gamma', \Gamma'' \vdash A}$,

Proof substitution

Proposition

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we can construct a proof $\dfrac{\pi[\pi'/A]}{\Gamma, \Gamma' \vdash B}$.

Proof.
By induction on π:

- we replace axioms $\dfrac{}{\Gamma, A, \Gamma', \Gamma'' \vdash A}\ (\text{ax})$ by $\dfrac{w(\pi')}{\Gamma, \Gamma', \Gamma'' \vdash A}$,
- $\dfrac{\dfrac{\pi_1}{\Gamma, A, \Gamma' \vdash B} \qquad \dfrac{\pi_2}{\Gamma, A, \Gamma' \vdash C}}{\Gamma, A, \Gamma' \vdash B \wedge C}$ becomes $\dfrac{\dfrac{\pi_1[\pi'/A]}{\Gamma, \Gamma' \vdash B} \qquad \dfrac{\pi_2[\pi'/A]}{\Gamma, \Gamma' \vdash C}}{\Gamma, \Gamma' \vdash B \wedge C}$, etc. ☐

Part IV

Cut elimination

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In mathematics, one often uses lemmas to show results.

Theorem
$\exists x . x + x = 4$.

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Theorem
$\exists x.x + x = 4$.

Proof.

- Lemma: every even number y admits a half:

$$\forall y.\,\mathsf{even}(y) \Rightarrow \exists x.x + x = y$$

- 4 is even. □

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In mathematics, one often uses lemmas to show results.

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Proof.
Take $x = 2$. ☐

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- Lemma: every even number y admits a half:

$$\forall y.\, \mathrm{even}(y) \Rightarrow \exists x. x + x = y$$

- 4 is even. $\square$

The proof of the lemma must certainly contain a way to compute the value for x.

This process of extraction is called **cut elimination**.

Cut elimination

Of course, this largely depends on the way we formalized things.

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We define even natural numbers as the smallest set $\mathsf{Even} \subseteq \mathbb{N}$ such that

- $0 \in \mathsf{Even}$,
- if $n \in \mathsf{Even}$ then $n + 2 \in \mathsf{Even}$.

Cut elimination

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We can then show that every even number y admits a half x by recurrence.

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- If $y = 0$ then we take $x = 0$ since $0 + 0 = 0$.

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- $0 \in \mathsf{Even}$,
- if $n \in \mathsf{Even}$ then $n + 2 \in \mathsf{Even}$.

We can then show that every even number y admits a half x by recurrence.

- If $y = 0$ then we take $x = 0$ since $0 + 0 = 0$.
- Otherwise $y = y' + 2$ for some even number y'.
  By recurrence, y' admits a half x' and we take $x = x' + 1$.
  Namely,

$$x + x = (x' + 1) + (x' + 1) = (x' + x') + 2 = y' + 2 = y$$

Cut elimination

Of course, this largely depends on the way we formalized things.

We define even natural numbers as the smallest set $\text{Even} \subseteq \mathbb{N}$ such that

- $0 \in \text{Even}$,
- if $n \in \text{Even}$ then $n + 2 \in \text{Even}$.

We can also show that 4 is even:

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We can also show that 4 is even:

- 0 is even,
- thus 2 is even,
- thus 4 is even.

And therefore 4 admits a half.

Cut elimination

From our proof, we can compute the half of 4:

- 4 is even, because 2 is even, because 0 is even,
- $\mathsf{half}(4) = \mathsf{half}(2) + 1 = (\mathsf{half}(0) + 1) + 1 = 0 + 1 + 1 = 2$

Cut elimination “reverse engineers” the proof in order to extract this witness.

Cuts

A **cut** is an elimination rule whose principal premise is proved by an introduction rule.

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma \vdash A} \qquad \dfrac{\pi'}{\Gamma \vdash B}}{\Gamma \vdash A \wedge B}\,(\wedge_{\mathsf{I}})}{\Gamma \vdash A}\,(\wedge^{\mathsf{l}}_{\mathsf{E}}) \qquad\qquad \dfrac{\dfrac{\dfrac{\pi}{\Gamma, A \vdash B}}{\Gamma \vdash A \Rightarrow B}\,(\Rightarrow_{\mathsf{I}}) \qquad \dfrac{\pi'}{\Gamma \vdash A}}{\Gamma \vdash B}\,(\Rightarrow_{\mathsf{E}})$$

A **cut-free** proof is a proof without cuts.

Cuts

An example of a proof with a cut is

$$A \vdash A$$

Cuts

An example of a proof with a cut is

$$\frac{A \vdash A \vee A \qquad A, A \vdash A \qquad A, A \vdash A}{A \vdash A}\ (\vee_E)$$

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$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,(\mathrm{ax})}{A \vdash A \vee A}\,(\vee^{\mathrm{l}}_{\mathrm{I}}) \qquad \dfrac{}{A, A \vdash A}\,(\mathrm{ax}) \qquad \dfrac{}{A, A \vdash A}\,(\mathrm{ax})}{A \vdash A}\,(\vee_{\mathrm{E}})$$

We can remove the cut and reduce it to

$$\dfrac{}{A \vdash A}\,(\mathrm{ax})$$

Cut elimination

Theorem
If a sequent is provable then it has a cut-free proof.

Proof.
The idea is to iteratively transform the proof of the original sequent in order to remove all cuts. □

Cut elimination: conjunctions

We can eliminate cuts on $\wedge$:

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma \vdash A} \qquad \dfrac{\pi'}{\Gamma \vdash B}}{\Gamma \vdash A \wedge B}\,(\wedge_{\mathrm{I}})}{\Gamma \vdash A}\,(\wedge_{\mathrm{E}}^{\mathrm{l}}) \qquad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma \vdash A} \qquad \dfrac{\pi'}{\Gamma \vdash B}}{\Gamma \vdash A \wedge B}\,(\wedge_{\mathrm{I}})}{\Gamma \vdash B}\,(\wedge_{\mathrm{E}}^{\mathrm{r}}) \qquad \rightsquigarrow$$

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Cut elimination: implications

We can eliminate cuts on $\Rightarrow$:

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma, A \vdash B}}{\Gamma \vdash A \Rightarrow B}\,(\Rightarrow_{\mathrm{I}}) \qquad \dfrac{\pi'}{\Gamma \vdash A}}{\Gamma \vdash B}\,(\Rightarrow_{\mathrm{E}}) \qquad \rightsquigarrow$$

Cut elimination: implications

We can eliminate cuts on $\Rightarrow$:

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma, A \vdash B}}{\Gamma \vdash A \Rightarrow B}\,(\Rightarrow_{\mathrm{I}}) \qquad \dfrac{\pi'}{\Gamma \vdash A}}{\Gamma \vdash B}\,(\Rightarrow_{\mathrm{E}}) \qquad \rightsquigarrow \qquad \dfrac{\pi[\pi'/A]}{\Gamma \vdash B}$$

where $\pi[\pi'/A]$ is π where we have replaced all axioms on A

$$\dfrac{}{\Gamma, A, \Gamma' \vdash A}\,(\mathrm{ax}) \qquad \text{by} \qquad \dfrac{w(\pi')}{\Gamma, A, \Gamma' \vdash A}$$

where $w(\pi')$ is an appropriate weakening of π.

Cut elimination: disjunctions

We can eliminate cuts on $\vee$:

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma \vdash A}}{\Gamma \vdash A \vee B}\,(\vee_{\mathsf{I}}^{\mathsf{l}}) \qquad \dfrac{\pi'}{\Gamma, A \vdash C} \qquad \dfrac{\pi''}{\Gamma, B \vdash C}}{\Gamma \vdash C}\,(\vee_{\mathsf{E}}) \qquad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma \vdash B}}{\Gamma \vdash A \vee B}\,(\vee_{\mathsf{I}}^{\mathsf{r}}) \qquad \dfrac{\pi'}{\Gamma, A \vdash C} \qquad \dfrac{\pi''}{\Gamma, B \vdash C}}{\Gamma \vdash C}\,(\vee_{\mathsf{E}}) \qquad \rightsquigarrow$$

Cut elimination: disjunctions

We can eliminate cuts on $\lor$:

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma \vdash A}}{\Gamma \vdash A \lor B}\,(\lor^{\mathsf{l}}_{\mathsf{I}}) \qquad \dfrac{\pi'}{\Gamma, A \vdash C} \qquad \dfrac{\pi''}{\Gamma, B \vdash C}}{\Gamma \vdash C}\,(\lor_{\mathsf{E}}) \quad \rightsquigarrow \quad \dfrac{\pi'[\pi/A]}{\Gamma \vdash C}$$

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma \vdash B}}{\Gamma \vdash A \lor B}\,(\lor^{\mathsf{r}}_{\mathsf{I}}) \qquad \dfrac{\pi'}{\Gamma, A \vdash C} \qquad \dfrac{\pi''}{\Gamma, B \vdash C}}{\Gamma \vdash C}\,(\lor_{\mathsf{E}}) \quad \rightsquigarrow \quad \dfrac{\pi''[\pi/B]}{\Gamma \vdash C}$$

Cut elimination: termination

For instance,

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, A \vdash A}\,(\text{ax}) \qquad \dfrac{}{\Gamma, A \vdash A}\,(\text{ax})}{\Gamma, A \vdash A \wedge A}\,(\wedge_{\text{I}})}{\Gamma \vdash A \Rightarrow A \wedge A}\,(\Rightarrow_{\text{I}}) \qquad \dfrac{\pi}{\Gamma \vdash A}}{\Gamma \vdash A \wedge A}\,(\Rightarrow_{\text{E}})$$

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is transformed into

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Note that if π contained n cuts then we now have $2n$ cuts...

Cut elimination: termination

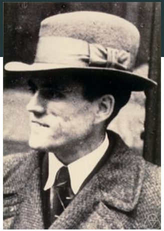
It is true that every order we chose to remove the cuts, we will end up on a cut-free proof after a finite number of steps, but it is quite difficult to show.

Cut elimination: termination

It is true that every order we chose to remove the cuts, we will end up on a cut-free proof after a finite number of steps, but it is quite difficult to show.

However, here, we only need to show that *a particular way* of eliminating cuts will terminates and this is not too hard (more on this if we have time).

Cut elimination



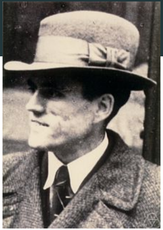
This cut elimination procedure is due to Gentzen (1930s).

We will see that it corresponds to "executing" a proof.

Or, as Girard put it:

> *A logic without cut elimination is like a car without an engine.*

Cut elimination



This cut elimination procedure is due to Gentzen (1930s).

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Or, as Girard put it:

> *A logic without cut elimination is like a car without an engine.*

Note that

- the cut-free proof is “simpler”,
- but it can be much bigger.

Think of a program computing the factorial of 1000.

Cut elimination: an important consequence

Proposition
A cut-free proof of $\vdash A$ *necessarily ends with an introduction rule.*

Proof.
We reason by induction on the proof. Consider the last rule of the proof of $\vdash A$:

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We reason by induction on the proof. Consider the last rule of the proof of $\vdash A$:

- it cannot be an axiom because the context is empty,
- if the last rule was an elimination rule, then by induction hypothesis the proof of the principal premise should end on an introduction rule and we would have a cut, e.g.

$$\frac{\vdash B \Rightarrow A \qquad \vdash B}{\vdash A}\ (\Rightarrow_{\mathsf{E}})$$

  contradiction,

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  $$\dfrac{\dfrac{\vdots}{\vdash B \Rightarrow A}\,(\Rightarrow_{\mathrm{I}}) \qquad \vdash B}{\vdash A}\,(\Rightarrow_{\mathrm{E}})$$

  contradiction,
- therefore the last rule is an introduction rule. □

Cut elimination: an important consequence

Proposition
A cut-free proof of $\vdash A$ *necessarily ends with an introduction rule.*

By the cut elimination theorem, every provable formula A admits a proof ending by an introduction rule.

Cut elimination: an important consequence

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A cut-free proof of $\vdash A$ *necessarily ends with an introduction rule.*

Note that a cut-free proof of $\Gamma \vdash A$ does not necessarily end with an introduction rule (with Γ non-empty):

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A cut-free proof of $\vdash A$ *necessarily ends with an introduction rule.*

Note that a cut-free proof of $\Gamma \vdash A$ does not necessarily end with an introduction rule (with Γ non-empty): what should be the introduction rule for

$$A \lor B \vdash A \lor B$$

Consistency

An easy way to implement a proof checker:

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let is_valid_proof p = true
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A logical system is consistent if and only if the formula $\bot$ *is not provable.*

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Proposition
A logical system is consistent if and only if the formula $\perp$ *is not provable.*

Proof.
If $\perp$ is not provable then the system is consistent.

Consistency

A logical system is **consistent** if there is at least one sequent which is not provable.

Proposition
A logical system is consistent if and only if the formula $\bot$ is not provable.

Proof.
If $\bot$ is not provable then the system is consistent.

If the system is consistent there is a non-provable sequent $\Gamma \vdash A$. If $\bot$ was provable we could deduce A:

$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash \bot}}{\Gamma \vdash \bot}\,(\mathrm{wk})}{\Gamma \vdash A}\,(\bot_{\mathrm{E}})$$

Contradiction. □

Consistency

Theorem
The logical system NJ is consistent.

Proof.
Suppose that $\perp$ is provable.

Consistency

Theorem
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Suppose that $\bot$ is provable.

By the cut elimination theorem, we would have a cut-free proof.

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By previous theorem, it would end with an introduction rule.

Consistency

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The logical system NJ is consistent.

Proof.
Suppose that $\bot$ is provable.

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By previous theorem, it would end with an introduction rule.

But there is no introduction rule for $\bot$.

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Contradiction. □

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By previous theorem, it would end with an introduction rule.

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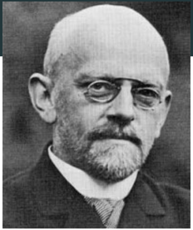
Contradiction. $\square$

Remark
Another way to formulate consistency:

there is no formula A such that both A and $\neg A$ are provable.

Consistency



At the beginning of the 20th century, Hilbert wanted to construct a collection of axioms which one could use as foundations of mathematics.

Since we are at day one, there is no semantics yet (= pre-existing mathematical world), everything is purely syntactic.

Consistency is thus one of the only possible correctness criterion for such an axiomatics.

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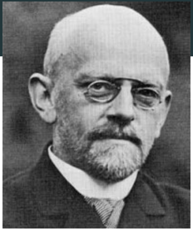

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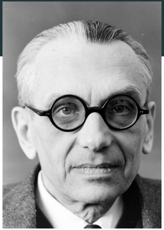

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But we cannot use “meta-mathematics” to prove consistency: we should therefore look for a system proving its own consistency.

Gödel’s second incompleteness theorem: this is impossible for a non-trivial system.

Commutative cuts

Cut elimination does not simplify all the proofs: we would like

$$\bot \vdash A$$

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$$\dfrac{\bot \vdash A \vee A \qquad\qquad \bot, A \vdash A \qquad\qquad \bot, A \vdash A}{\bot \vdash A}\ (\vee_{\mathrm{E}})$$

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to simplify into

$$\dfrac{\dfrac{}{\bot \vdash \bot}\,(\mathrm{ax})}{\bot \vdash A}\,(\bot_{\mathrm{E}})$$

Commutative cuts

A typical cut is

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,(\mathrm{ax}) \qquad \dfrac{}{A \vdash A}\,(\mathrm{ax})}{A \vdash A \land A}\,(\land_{\mathrm{I}})}{A \vdash A}\,(\land_{\mathrm{E}}^{\mathrm{l}})$$

Commutative cuts

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Some "cuts" are "too far apart" to be eliminated:

$$\dfrac{\dfrac{\dfrac{}{\ldots \vdash B \vee C}\,(\mathrm{ax}) \qquad \dfrac{\dfrac{}{\ldots \vdash A}\,(\mathrm{ax}) \qquad \dfrac{}{\ldots \vdash A}\,(\mathrm{ax})}{A, B \vee C, B \vdash A \wedge A}\,(\wedge_{\mathrm{I}}) \qquad \dfrac{\dfrac{}{\ldots \vdash A}\,(\mathrm{ax}) \qquad \dfrac{}{\ldots \vdash A}\,(\mathrm{ax})}{A, B \vee C, C \vdash A \wedge A}\,(\wedge_{\mathrm{I}})}{A, B \vee C \vdash A \wedge A}\,(\vee_{\mathrm{E}})}{A, B \vee C \vdash A}\,(\wedge^{\mathrm{l}}_{\mathrm{E}})$$

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This is due to the form of the rule

$$\dfrac{\Gamma \vdash A \vee B \qquad \Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma \vdash C}\,(\vee_{\mathrm{E}})$$

where the C comes out of nowhere.

Commutative cuts

Those can be handled by adding the following elimination rules for **commutative cuts**:

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma \vdash A \lor B} \qquad \dfrac{\pi'}{\Gamma, A \vdash C} \qquad \dfrac{\pi''}{\Gamma, B \vdash C}}{\Gamma \vdash C}\,(\lor_{\mathsf{E}}) \qquad \ldots}{\Gamma \vdash D}\,(?_{\mathsf{E}})$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\pi}{\Gamma \vdash A \lor B} \qquad \dfrac{\dfrac{\pi'}{\Gamma, A \vdash C} \qquad \ldots}{\Gamma, A \vdash D}\,(?_{\mathsf{E}}) \qquad \dfrac{\dfrac{\pi''}{\Gamma, B \vdash C} \qquad \ldots}{\Gamma, B \vdash D}\,(?_{\mathsf{E}})}{\Gamma \vdash D}\,(\lor_{\mathsf{E}})$$

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Those can be handled by adding the following elimination rules for **commutative cuts**:

$$\dfrac{\dfrac{\Gamma \vdash \bot}{\Gamma \vdash A}\,(\bot_{\mathsf{E}}) \qquad \ldots}{\Gamma \vdash B}\,(?_{\mathsf{E}}) \quad \rightsquigarrow \quad \dfrac{\Gamma \vdash \bot}{\Gamma \vdash B}\,(\bot_{\mathsf{E}})$$

Part V

Intuitionism

Proving disjunctions

The following is in accordance to the intuitionistic point of view:

Proposition
If $A \vee B$ is provable then either A or B is provable.

Proving disjunctions

The following is in accordance to the intuitionistic point of view:

Proposition
If $A \lor B$ is provable then either A or B is provable.

Proof.
There is a cut-free proof, which begins by an introduction rule:

$$\dfrac{\dfrac{\pi}{\vdash A}}{\vdash A \lor B}\;(\lor_{i}^{l}) \qquad \text{or} \qquad \dfrac{\dfrac{\pi}{\vdash B}}{\vdash A \lor B}\;(\lor_{i}^{r})$$

and therefore we have a proof of A or a proof of B. □

Excluded middle

An interesting consequence is that

Proposition
The law of ***excluded middle*** $X \vee \neg X$ *is not provable.*

Proof.
If it was the case we would have either

- A proof of X, but there is no introduction rule

$$\frac{\cdots}{\vdash X}\ (?_{\mathrm{I}})$$

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The law of ***excluded middle*** $X \lor \neg X$ *is not provable.*

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If it was the case we would have either

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  By substitution, π should induce a proof $\dfrac{\vdots}{\top \vdash \bot}$

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- A proof of $\neg X$, which should begin with the introduction rule $\dfrac{\dfrac{\pi}{X \vdash \bot}}{\vdash \neg X}\ (\neg_{\mathrm{I}})$

  By substitution, π should induce a proof $\dfrac{\vdots}{\top \vdash \bot}$

  By $\top$-strengthening, we thus have a proof $\dfrac{\vdots}{\vdash \bot}$

Excluded middle

An interesting consequence is that

Proposition
*The law of **excluded middle** $X \vee \neg X$ is not provable.*

Proof.
If it was the case we would have either

- A proof of $\neg X$, which should begin with the introduction rule $\dfrac{\dfrac{\pi}{X \vdash \bot}}{\vdash \neg X}\ (\neg_{\mathrm{I}})$

  By substitution, π should induce a proof $\dfrac{\vdots}{\top \vdash \bot}$

  By $\top$-strengthening, we thus have a proof $\dfrac{\vdots}{\vdash \bot}$

  But there is no introduction rule for $\bot$, contradiction. ☐

Excluded middle

Proposition
The law of ***excluded middle*** $X \vee \neg X$ *is not provable.*

Of course this does not say that, for a given value of X, we cannot decide:

- for $X = \top$, X is provable,
- for $X = \bot$, $\neg X$ is provable,
- for $X = Y \Rightarrow Y$, X provable,
- etc.

but we cannot do it in a generic way.

Excluded middle

Since $X \vee \neg X$ is not provable, maybe is it false?

Excluded middle

Since $X \vee \neg X$ is not provable, maybe is it false? No.

Proposition
The formula $\neg(X \vee \neg X)$ *cannot be proved.*

Proof.
We can prove $\neg\neg(X \vee \neg X)$, see next slide.

Excluded middle

Since $X \vee \neg X$ is not provable, maybe is it false? No.

Proposition
The formula $\neg(X \vee \neg X)$ *cannot be proved.*

Proof.
We can prove $\neg\neg(X \vee \neg X)$, see next slide.

If $\neg(X \vee \neg X)$ was also provable, we would have

$$\dfrac{\dfrac{\vdots}{\vdash \neg\neg(X \vee \neg X)} \qquad \dfrac{\vdots}{\vdash \neg(X \vee \neg X)}}{\vdash \bot}\,(\neg_{\mathrm{E}})$$

which is excluded since our logic is consistent. $\square$

Excluded middle

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$\vdash \neg(\neg X \vee X)$}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$\neg(\neg X \vee X), X \vdash \neg(\neg X \vee X)$}
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\RightLabel{$(\vee_{\mathsf{I}}^{\mathsf{r}})$}
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Intuitionistic vs classical logic

We have two logics:

- **intuitionistic logic**: formulas provable in NJ,
- **classical logic**: formulas valid in boolean models.

Intuitionistic vs classical logic

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- **intuitionistic logic**: formulas provable in NJ,
- **classical logic**: formulas valid in boolean models.

Proposition
A formula which is intuitionistically provable is classically valid.

But the converse is not true: $X \lor \neg X$ is classically valid but not provable.

Classical logic: NK

We call NK the system obtained from NJ by adding the axiom

$$\frac{}{\Gamma \vdash A \vee \neg A}\ (\mathrm{lem})$$

Theorem
A formula is valid in boolean models if and only if it can be proved in NK.

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We call NK the system obtained from NJ by adding the axiom

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Theorem
A formula is valid in boolean models if and only if it can be proved in NK.

Another possible axiom is

$$\dfrac{}{\Gamma \vdash \neg\neg A \Rightarrow A}$$

Exercise
Show this.

Classical logic

A typical classical reasoning:

Proposition
There exist two irrational numbers a and b such that a^b is rational.

Proof.

We know that $\sqrt{2}$ is irrational (if $\sqrt{2} = p/q$ then $p^2 = 2q^2$, prime factors).

The number ${\sqrt{2}}^{\sqrt{2}}$ is either rational or irrational.

- If it is rational, we are done.
- Otherwise, take $a = {\sqrt{2}}^{\sqrt{2}}$ and $b = \sqrt{2}$, we have $a^b = 2$. ☐

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There exist two irrational numbers a *and* b *such that* a^b *is rational.*

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- If it is rational, we are done.
- Otherwise, take $a = {\sqrt{2}}^{\sqrt{2}}$ and $b = \sqrt{2}$, we have $a^b = 2$. □

Classical logic is not **constructive**: we don't know which one.

Classical logic

Another typical reasoning:

Proposition
For every program p, either it stops after a finite number of steps or it does not.

However, no program can solve the **halting problem**.

Classical logic

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Proposition
For every program p, either it stops after a finite number of steps or it does not.

However, no program can solve the **halting problem**.

As a variant, do we really want to accept the following?

$$\forall x \in \mathbb{R}. x = 0 \lor x \neq 0$$

Classical logic

Intuitionism does not validate either

$$\neg\neg X \Rightarrow X$$

Classical logic

Intuitionism does not validate either

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I don't have lost my key. Where is it?

Classical logic

Proposition
Either $(\pi + e)$ *or* $(\pi - e)$ *is irrational.*

Proof.
Suppose that both $(\pi + e)$ and $(\pi - e)$ are irrational. Therefore

$$2\pi = (\pi + e) + (\pi - e)$$

is irrational. Contradiction (we know that π is irrational). □

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Logically, we have shown $\neg A \Rightarrow \bot$ and deduced A, i.e. used $\neg\neg A \Rightarrow A$.

Classical logic: semantical intuition

As before, we interpret formulas:

- $[\![A]\!]$ is a set (of “proofs” of A)
- $[\![A \Rightarrow B]\!] = [\![A]\!] \to [\![B]\!]$ is the set of functions
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Thus,

- if $[\![A]\!] = \emptyset$ then $[\![\neg\neg A]\!] = \emptyset$,
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Classical logic: double negation translation

Can we really prove more in classical logic?

Theorem (Glivenko 1929)
A sequent $\Gamma \vdash A$ *is classically provable if and only if* $\neg\neg\Gamma \vdash \neg\neg A$ *is intuitionistically provable.*

where $\neg\neg\Gamma$ means double negating every formula in Γ.

Classical logic: double negation translation

Theorem
Intuitionistic logic is consistent iff classical logic is consistent.

Proof.

Classical logic: double negation translation

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If classical logic is inconsistent, we have a classical proof of $\bot$ and thus an intuitionistic proof of $\neg\neg\bot$. However, the implication $\neg\neg\bot \Rightarrow \bot$ holds intuitionistically:

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$$\frac{\neg\neg\bot \vdash \bot}{\vdash \neg\neg\bot \Rightarrow \bot}\ (\Rightarrow_{\mathrm{I}})$$

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$$\dfrac{\dfrac{\neg\neg\bot \vdash \neg\neg\bot \qquad\qquad \neg\neg\bot \vdash \neg\bot}{\neg\neg\bot \vdash \bot}\,(\neg_{\mathrm{E}})}{\vdash \neg\neg\bot \Rightarrow \bot}\,(\Rightarrow_{\mathrm{I}})$$

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Proof.
If classical logic is inconsistent, we have a classical proof of $\bot$ and thus an intuitionistic proof of $\neg\neg\bot$. However, the implication $\neg\neg\bot \Rightarrow \bot$ holds intuitionistically. We thus have an intuitionistic proof of $\bot$:

$$\dfrac{\dfrac{\vdots}{\vdash \neg\neg\bot \Rightarrow \bot} \qquad \dfrac{\pi}{\vdash \neg\neg\bot}}{\vdash \bot}\ (\Rightarrow_{\mathsf{E}})$$

□

Classical logic: double negation translation

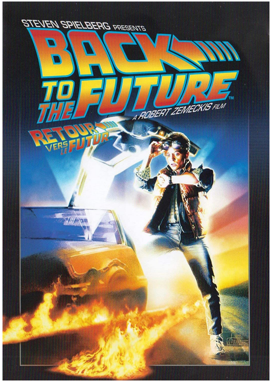
What about adding even more negations?

Lemma
For $n > 0$, we have $\neg^{n+2} A \Leftrightarrow \neg^n A$.

Proof.
The implications $A \Rightarrow \neg\neg A$ and $\neg\neg\neg A \Rightarrow A$ are intuitionistically provable. □

Classical logic: operational intuition

Classical logic allows to “go back in time” and change our decisions based on current knowledge.



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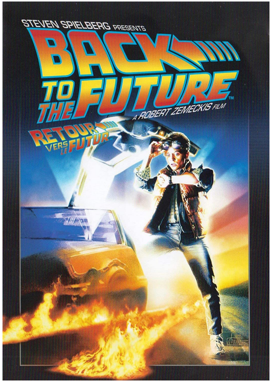

Use this to show $A \vee (A \Rightarrow \bot)$.

Classical logic: formal operational intuition

Let us formalize this intuition by showing that the rule

$$\frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A}\ (\neg\neg_{\mathsf{E}})$$

allows us to derive

$$\neg A \vee A$$

Excluded middle from double negation elimination

$$\vdash \neg A \vee A$$

Excluded middle from double negation elimination

$$\dfrac{\vdash \neg\neg(\neg A \vee A)}{\vdash \neg A \vee A}\,(\neg\neg_{\mathsf{E}})$$

Excluded middle from double negation elimination

$$\dfrac{\dfrac{\neg(\neg A \lor A) \vdash \bot}{\vdash \neg\neg(\neg A \lor A)}\;(\neg_{\mathsf{I}})}{\vdash \neg A \lor A}\;(\neg\neg_{\mathsf{E}})$$

Excluded middle from double negation elimination

$$\dfrac{\dfrac{\dfrac{\neg A \lor A) \qquad \neg(\neg A \lor A) \vdash \neg A \lor A}{\neg(\neg A \lor A) \vdash \bot}\,(\neg_E)}{\vdash \neg\neg(\neg A \lor A)}\,(\neg_I)}{\vdash \neg A \lor A}\,(\neg\neg_E)$$

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$$
\dfrac{
  \dfrac{
    \dfrac{}{\neg A \vee A)}\,(\mathsf{ax})
    \qquad
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{}{\neg(\neg A \vee A), A \vdash \neg(\neg A \vee A)}\,(\mathsf{ax})
          \qquad
          \dfrac{\neg(\neg A \vee A), A \vdash A}{\neg(\neg A \vee A), A \vdash \neg A \vee A}\,(\vee_{\mathsf{I}}^{\mathsf{r}})
        }{\neg(\neg A \vee A), A \vdash \bot}\,(\neg_{\mathsf{E}})
      }{\neg(\neg A \vee A) \vdash \neg A}\,(\neg_{\mathsf{I}})
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More generally, we have proofs of the form

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$$

Intermediate logics

Are there intermediate principles between intuitionistic and classical?

Intermediate logics

Are there intermediate principles between intuitionistic and classical?

Yes: for instance,

$$\neg\neg A \vee \neg A$$

or

$$(A \Rightarrow B) \vee (B \Rightarrow A)$$

Part VI

Termination of cut elimination

Termination of cut elimination

Remember that cut elimination was obtained by performing some transformations on proofs:

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma \vdash A} \qquad \dfrac{\pi'}{\Gamma \vdash B}}{\Gamma \vdash A \wedge B}\,(\wedge_\mathsf{I})}{\Gamma \vdash A}\,(\wedge_\mathsf{E}^\mathsf{l}) \quad \rightsquigarrow \quad \dfrac{\pi}{\Gamma \vdash A}$$

What we still have to show is that this process terminates at some point.

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What we still have to show is that this process terminates at some point.

Intuitively, this can be achieved by associating a *size* (e.g. in $\mathbb{N}$) to each proof so that each reduction step is strictly decreasing:

$$\begin{array}{ccccccc} \pi_0 & \rightsquigarrow & \pi_1 & \rightsquigarrow & \pi_2 & \rightsquigarrow & \ldots \\ n_0 & > & n_1 & > & n_2 & > & \ldots \end{array}$$

First try

Based on the previous rule, we could say that the size of a proof is the number of rules we use:

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma \vdash A} \qquad \dfrac{\pi'}{\Gamma \vdash B}}{\Gamma \vdash A \wedge B}\,(\wedge_{\mathsf{I}})}{\Gamma \vdash A}\,(\wedge_{\mathsf{E}}^{\mathsf{l}}) \qquad \rightsquigarrow \qquad \dfrac{\pi}{\Gamma \vdash A}$$

$$n_{\pi} + n_{\pi'} + 2 \qquad > \qquad n_{\pi}$$

First try

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$$n_\pi + 5 \qquad > \qquad 2n_\pi + 1$$

This does not work (and neither does the number of cuts).

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$$n_\pi + 5 \qquad > \qquad 2n_\pi + 1$$

This does not work (and neither does the number of cuts).

We have more cuts, but they are smaller!

We have to come up with a much more subtle notion of size.

Size of formulas

The **size** $|A|$ of a formula A is its number of connectives:

$$
\begin{aligned}
|X| &= 1\\
|\top| &= 1\\
|\bot| &= 1\\
|A \Rightarrow B| &= 1 + |A| + |B|\\
|A \wedge B| &= 1 + |A| + |B|\\
|A \vee B| &= 1 + |A| + |B|
\end{aligned}
$$

Degree of a cut

The **degree** of a cut is the size of the cut formula.

For instance, the degree of

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma \vdash A} \qquad \dfrac{\pi'}{\Gamma \vdash B}}{\Gamma \vdash A \wedge B}\,(\wedge_\mathsf{I})}{\Gamma \vdash A}\,(\wedge_\mathsf{E}^\mathsf{l})$$

is

$$|A \wedge B| = 1 + |A| + |B|$$

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Formally, a **multiset** M of elements in X is a function

$$M : X \to \mathbb{N}$$

whose domain

$$\mathrm{dom}(M) = \{x \in X \mid M(x) \neq 0\}$$

is finite.

Degree of a proof

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We write $M \sqcup N$ for the (disjoint) union of multisets:

$$\{a, b, b\} \sqcup \{a, c\} = \{a, a, b, b, c\}$$

i.e.

$$(M \sqcup N)(x) = M(x) + N(x)$$

Ordering multisets

We order multisets of natural numbers by $M > M'$ whenever M' can be obtained from M by iteratively replacing an element by (multiple) strictly smaller elements:

$$\{4, 4, 3\} > \{4, 3, 3, 3, 1\} > \{4, 3, 3, 3\} > \{4, 3, 3, 2, 2\} > \dots$$

Theorem (Dershowitz-Manna)
There is no infinite strictly decreasing sequence of finite multisets.

Termination of cut elimination

Cut elimination makes the degree strictly decrease:

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma \vdash A} \qquad \dfrac{\pi'}{\Gamma \vdash B}}{\Gamma \vdash A \wedge B}\,(\wedge_{\mathsf{I}})}{\Gamma \vdash A}\,(\wedge_{\mathsf{E}}^{\mathsf{l}}) \qquad \rightsquigarrow \qquad \dfrac{\pi}{\Gamma \vdash A}$$

$$M_\pi \sqcup M_{\pi'} \sqcup \{|A \wedge B|\} \qquad > \qquad M_\pi$$

Termination of cut elimination

Cut elimination makes the degree strictly decrease:

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma, A \vdash B}}{\Gamma \vdash A \Rightarrow B}\,(\Rightarrow_{\mathsf{I}}) \qquad \dfrac{\pi'}{\Gamma \vdash A}}{\Gamma \vdash B}\,(\Rightarrow_{\mathsf{E}}) \qquad \rightsquigarrow \qquad \dfrac{\pi[\pi'/A]}{\Gamma \vdash B}$$

$$M_\pi \sqcup \{|A \Rightarrow B|\} \sqcup M_{\pi'} \qquad > \qquad M_\pi \sqcup \{|A|, |A|, \ldots\} \sqcup (n \times M_{\pi'})$$

and we fail!

Termination of cut elimination

Cut elimination makes the degree strictly decrease:

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma, A \vdash B}}{\Gamma \vdash A \Rightarrow B}\,(\Rightarrow_{\mathsf{I}}) \qquad \dfrac{\pi'}{\Gamma \vdash A}}{\Gamma \vdash B}\,(\Rightarrow_{\mathsf{E}}) \qquad \rightsquigarrow \qquad \dfrac{\pi[\pi'/A]}{\Gamma \vdash B}$$

$$M_\pi \sqcup \{|A \Rightarrow B|\} \qquad > \qquad M_\pi \sqcup \{|A|, |A|, \ldots\}$$

If we eliminate π' of highest depth we can suppose $M_{\pi'} = \emptyset$!

Termination of cut elimination

For instance, we replace one cut on $A \Rightarrow A \land A$ by two on A:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, A \vdash A}\,(\text{ax}) \qquad \dfrac{}{\Gamma, A \vdash A}\,(\text{ax})}{\Gamma, A \vdash A \land A}\,(\land_{\text{I}})}{\Gamma \vdash A \Rightarrow A \land A}\,(\Rightarrow_{\text{I}}) \qquad \dfrac{\pi}{\Gamma \vdash A}}{\Gamma \vdash A \land A}\,(\Rightarrow_{\text{E}}) \quad \rightsquigarrow \quad \dfrac{\dfrac{\pi}{\Gamma \vdash A} \qquad \dfrac{\pi}{\Gamma \vdash A}}{\Gamma \vdash A \land A}\,(\land_{\text{I}})$$

$$M_\pi \sqcup \{|A \Rightarrow A \land A|\} \qquad > \qquad M_\pi \sqcup \{|A|, |A|\}$$

Part VII

Kripke semantics

The completeness theorem

We have recalled the **completeness theorem** for classical logic:

Theorem
A formula is classically provable if and only if it is valid, i.e. its interpretation is true for every valuation.

Can we have a semantics with the same properties for intuitionistic logic?

Kripke structures

A **Kripke structure** $(W, \leqslant, \rho)$ consists of

- a set W of *worlds*,
- a partial order $\leqslant$,
- a function $\rho : W \to \mathcal{X} \to \{0, 1\}$

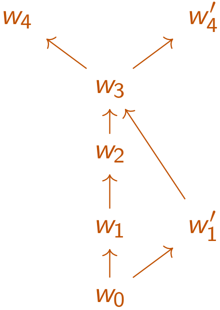
such that for every $w, w' \in W$

$$w \leqslant w' \text{ and } \rho(w, x) = 1 \qquad \text{implies} \qquad \rho(w', x) = 1$$

Kripke structures

To every world w, the function ρ associates a valuation $\rho w : \mathcal{X} \to \{0, 1\}$.

We can think of $w \leqslant w'$ as w' being a **future** of w:



and the condition says:

what is true now will always be true in the future.

Satisfaction

Given a Kripke structure W, a formula A is **satisfied** in a world w, written

$$w \vDash_W A$$

when

$$\begin{array}{ll}
w \vDash X & \text{iff } \rho(w, X) \\
w \vDash \top & \text{holds} \\
w \vDash \bot & \text{does not hold} \\
w \vDash A \wedge B & \text{iff } w \vDash A \text{ and } w \vDash B \\
w \vDash A \vee B & \text{iff } w \vDash A \text{ or } w \vDash B \\
w \vDash A \Rightarrow B & \text{iff, for every } w' \geqslant w,\ w' \vDash A \text{ implies } w' \vDash B
\end{array}$$

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What is $w \vDash \neg A = A \Rightarrow \bot$?

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What is $w \vDash \neg A = A \Rightarrow \bot$?

$$w \vDash \neg A \text{ iff, for every } w' \geqslant w,\ w' \vDash A \text{ does not hold}$$

Validity

Given a context $\Gamma = A_i$, a formula A and a Kripke structure A, we write

$$\Gamma \vDash_W A$$

when for every $w \in W$, if $w \vDash_W A_i$ for each i, then $w \vDash_W A$.

We write $\Gamma \vDash A$ when $\Gamma \vDash_W A$ for every Kripke structure W and say that A is **valid** in the context Γ.

Soundness

Theorem
If $\Gamma \vdash A$ *is derivable intuitionistically then* $\Gamma \vDash A$*.*

Proof.
By induction. □

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By induction. ☐

By contraposition:

- if $\Gamma \nvDash A$
- if, for some structure W, $\Gamma \nvDash_W A$
- if, for some structure W and some w, $w \vDash A_i$ for all i and $w \nvDash_W A$

then $\Gamma \vdash A$ is not derivable.

An unprovable formula

Let's show that $\neg X \vee X$ is not provable.

Consider

with $w_0 \not\models X$ and $w_1 \models X$.

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We have $w_0 \nvDash \neg X$.

An unprovable formula

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Therefore $w_0 \nvDash \neg X \vee X$.

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w_0

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We have $w_0 \not\models \neg X$ and $w_1 \neg \models \neg X$, thus $w_0 \models \neg\neg X$.

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We have $w_0 \not\models \neg X$ and $w_1 \neg \models \neg X$, thus $w_0 \models \neg\neg X$.

But $w_0 \not\models X$, thus $w_0 \not\models \neg\neg X \Rightarrow X$.

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Then $\neg X$ means “I have lost my key”: I will never find it again in the future.

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Then $\neg X$ means “I have lost my key”: I will never find it again in the future.

Then $\neg\neg X \Rightarrow X$ does not hold: it is not true that if I have not lost my key, I have my key (e.g. it could be somewhere in my apartment!).

Completeness

The converse property also holds:

Theorem
$\Gamma \vdash A$ *is derivable intuitionistically iff* $\Gamma \vDash A$*.*