Higher dimensional automata

between topology and concurrency

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GETCO 22

Goals

- Introduce precubical sets alias higher dimensional automata.
- Topological executions: directed path spaces.
- Combinatorial executions: track complexes.
- How these models are related?

Directed spaces

Idea

Model computer programs by topological spaces.

- points of space = states of a program,
- distinguished paths = (partial) executions.

Definition (Grandis)

A *d-space* is a pair $(X, \vec{P}(X))$, where

- X is a topological space,
- $\vec{P}(X) \subseteq P(X) =: map(I, X)$ is a family of *d-paths* ($I = [0, 1]$),
- $\forall_{x \in X}\ const_x \in \vec{P}(X)$,
- $\alpha, \beta \in \vec{P}(X),\ \alpha(1) = \beta(0) \implies \alpha * \beta \in \vec{P}(X)$.
- $\alpha \in \vec{P}(X),\ f : I \to I$ increasing $\implies \alpha \circ f \in \vec{P}(X)$.

A map $f : X \to Y$ between d-spaces is a *d-map* if $\alpha \in \vec{P}(X) \implies f(\alpha) \in \vec{P}(Y)$.

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Directed spaces: examples

Example

Directed interval: $\vec{I} = (I, \{\alpha : I \to I \text{ increasing}\})$.

The category **dTop** of d-spaces and d-maps is complete and cocomplete.
We obtain more examples:

Example

- *Directed cube*: $\vec{I}^n = (I^n, \{\alpha : I \to I^n \text{ all coordinates increasing}\})$

- *Directed circle*: $\vec{S}^1 = \vec{I}/0 \sim 1 = (S^1, \{\text{counterclockwise paths}\})$

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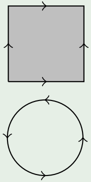

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Precubical sets

Definition

A *precubical set* K consist of

- a sequence of sets $(K[n])_{n\geq 0}$ (*n-cells* or *n-cubes*),
- a collection of maps $\delta_i^\varepsilon : K[n] \to K[n-1]$, $1 \leq i \leq n$, $\varepsilon = 0, 1$ (*face maps*),
- $\delta_i^\varepsilon \circ \delta_j^\eta = \delta_{j-1}^\eta \circ \delta_i^\varepsilon$ for $i < j$ (*precubical identities*).

A *precubical map* $f : K \to L$ is a sequence of compatible functions $f[n] : K[n] \to L[n]$.

Definition

The *geometric realization* of a precubical set K:

$$|K| = \coprod_{n\geq 0} K[n] \times \vec{I}^n / \sim$$

$$(\delta_i^\varepsilon(c), (x_1, \dots, x_{n-1})) \sim (c, (x_1, \dots, x_{i-1}, \varepsilon, x_i, \dots, x_{n-1})).$$

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Examples of precubical sets

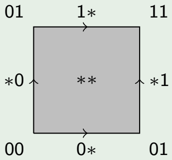
Example

The *standard n-cube* $\square^n$:

- $\square^n[k] = \{(a_1, \ldots, a_n) \mid a_i \in \{0, *, 1\}$, exactly k stars among a_i's$\}$.
- δ_i^ε converts i-th star into ε.

The geometric realization of $\square^n$ is $\vec{I}^n$.

A *Euclidean complex* is a precubical subset of a standard cube.



Example

The *final precubical set Z*:

- $Z[n]$ has exactly one element for every n,
- face maps are defined the only possible way.

Directed paths spaces on precubical sets

Question

Let K be a precubical set, $v, w \in K[0]$ its vertices.
What is the (homotopy type of) the space $\vec{P}(K)_v^w$ of directed paths in $|K|$ from v to w?

Results for Euclidean complexes

- $\vec{P}(\vec{I}^n)_0^1$ is contractible,
- $\vec{P}(\partial\vec{I}^n)_0^1 \simeq S^{n-2}$,
- The length decomposition: $\vec{P}(K)_v^w = \coprod_{n\geq 0} \vec{P}(K;n)_v^w$.
- Prodsimplicial models for Euclidean complexes [Raussen 2010, 2012].
- Every finite CW-complex is homotopy equivalent to $\vec{P}(K)_0^1$ for $K \subseteq \square^n$ [Z, 2016].

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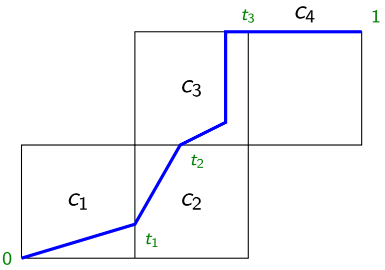
Presentations of d-paths

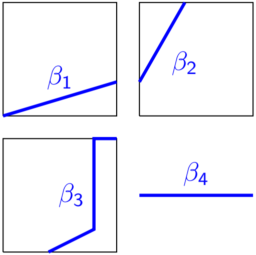
Observation

Every d-path $\alpha \in \vec{P}(K)$ has a presentation

$$\alpha = [c_1; \beta_1] \overset{t_1}{*} [c_2; \beta_1] \overset{t_2}{*} \cdots \overset{t_{n-1}}{*} [c_n; \beta_n]$$

where $c_k \in K[n_k]$, $\beta_k \in \vec{P}_{[t_{k-1},t_k]}(\vec{I}^{n_k})$, $0 < t_1 < \cdots < t_{n-1} < 1$.





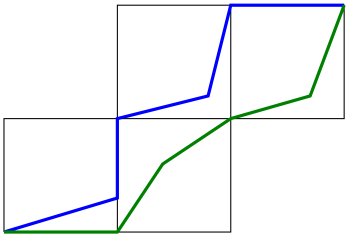
Tame paths

Definition

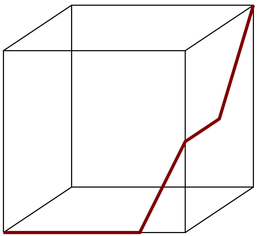
A d-path $\alpha \in \vec{P}(K)$ is *tame* if there exists a presentation

$$\alpha = [c_1; \beta_1] \overset{t_1}{*} \cdots \overset{t_{n-1}}{*} [c_n; \beta_n]$$

such that $\beta_k(t_{k-1}) = (0, \ldots, 0)$ and $\beta_k(t_k) = (1, \ldots, 1)$ for all k.



Both paths are tame



Tame in $\Box^3$ but not in $\partial\Box^3$.

Taming theorem

Theorem [Z 2020]

For every precubical set K, the inclusion

$$\vec{P}_{tame}(K)_v^w \subseteq \vec{P}(K)_v^w$$

is a homotopy equivalence.

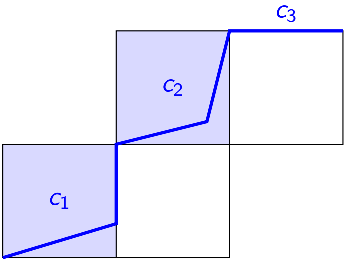
Cube chains

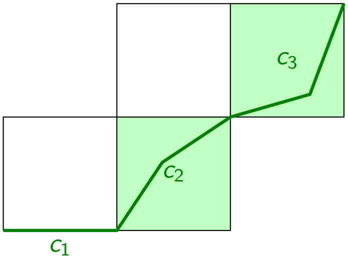
Definition

A *cube chain* in K from v to w is a sequence of cubes $(c_1, \dots, c_n)$, such that

$$\delta^0_{all}(c_1) = v, \quad \delta^1_{all}(c_k) = \delta^0_{all}(c_{k+1}), \quad \delta^1_{all}(c_n) = w.$$

Every tame path "lies" in a cube chain:





Cube chains on Euclidean complexes

Proposition

If K is a Euclidean complex, then:

- The set $Ch(K)_v^w$ of cube chains from v to w is a poset with respect to inclusion.
- The set of paths $\vec{P}(K; C)$ lying in cube chain C is contractible.
- $\bigcap_{j=1}^k \vec{P}(K; C_j)$ is contractible if there exists C' such that $C' \leq C_j$ for all j.
- Otherwise, $\bigcap_{j=1}^k \vec{P}(K; C_j)$ is empty.

Thus, $\{\vec{P}(K; C) \mid C \in Ch(K)_v^w\}$ is a good cover of $\vec{P}_{tame}(K)_v^w$.

Theorem [Z 2018]

If K is a Euclidean complex, then Nerve Lemma implies:

$$\vec{P}(K)_v^w \simeq \vec{P}_{tame}(K)_v^w \simeq |Ch(K)_v^w|.$$

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A model for directed paths on Euclidean complexes

Proposition

The following posets are isomorphic:

- $Ch(\square^n)_0^1$,
- The poset of ordered partitions of $\{1, \ldots, n\}$.
- The face lattice of $(n-1)$-dimensional permutahedron

$$\Pi^{n-1} = conv\{(\sigma(1), \ldots, \sigma(n)) \mid \sigma \in Perm(\{1, \ldots, n\})\}.$$

Example

If $K \subseteq \square^n$, then $|Ch(K)_0^1|$ is a subcomplex of the permutahedron, for example

$$|Ch(\partial\square^3)_0^1| =$$

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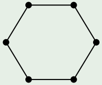


Applications

Algorithm

If $K \subseteq \square^n$ (or $K \subseteq [0,n_1] \times [0,n_2] \times \cdots \times [0,n_d]$), then there is an efficient algorithm for calculating $H_*(\vec{P}(K)_v^w)$ via discrete Morse theory.

Theorem (Raussen-Meshulam, Z)

Calculation of homology of $\vec{P}(K)_v^w$ for K being the k–skeleton of

$$[0,n_1] \times [0,n_2] \times \cdots \times [0,n_d].$$

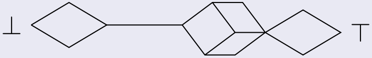
This is a "no $(k+1)$-equal" configuration spaces of sequences of points on $\mathbb{R}$.

Cube chain complex: general case

Definition

The *wedge cube* is a precubical set $\square^{\vee\mathbf{n}} = \square^{n_1} \underset{1\sim 0}{\vee} \cdots \underset{1\sim 0}{\vee} \square^{n_k}$. For example,

$$\square^{\vee(2,1,3,2)} = \square^2 \vee \square^1 \vee \square^3 \vee \square^1 =$$



The *cube chain* in K is a (bipointed) precubical map $\mathbf{c} : (\square^{\vee\mathbf{n}}, \bot, \top) \to (K, v, w)$.

Problem

If K is not a Euclidean complex, then a d-path may have different "tame" presentations using the same cube chain $\mathbf{c}$. As a consequence, the map

$$\mathbf{c}_* : \vec{P}(\square^{\vee\mathbf{n}})_\bot^\top \to \vec{P}_{tame}(K)_v^w$$

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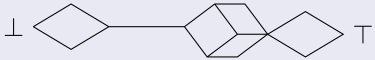

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Cube chain category

Definition

The *cube chain category* $Ch(K)_v^w$ of K:

- objects = cube chains $\mathbf{c} : (\square^{\vee\mathbf{n}}, \bot, \top) \to (K, v, w)$,
- morphisms = commutative diagrams of bipointed precubical maps

$$\begin{array}{ccc} \square^{\vee\mathbf{m}} & \searrow & \\ \downarrow & & K \\ \square^{\vee\mathbf{n}} & \nearrow & \end{array}$$

Theorem [Z 2020]

For every precubical set K there are homotopy equivalences

$$|Ch(K)_v^w| \simeq \vec{P}_{tame}(K)_v^w \simeq \vec{P}(K)_v^w.$$

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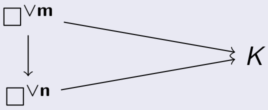

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Permutahedral complexes

Definition

The *cube wedge* category $\mathcal{P}$:

- objects: cube wedges $\square^{\vee \mathbf{n}} = \square^{n_1} \vee \cdots \vee \square^{n_k}$,
- morphisms: precubical maps preserving the initial and final vertices.

Properties

- For every bipointed precubical set K there is a forgetful functor

$$Ch(K)_v^w \ni (\mathbf{c} : \square^{\vee \mathbf{n}} \to K) \mapsto \square^{\vee \mathbf{n}} \in \mathcal{P}.$$

- The cube chains on K form a presheaf on $\mathcal{P}$ (a functor $\mathcal{P}^{op} \to \mathbf{Set}$):

$$Ch(K)(\square^{\vee \mathbf{n}}) = \square\mathbf{Set}_*^*(\square^{\vee \mathbf{n}}, K).$$

- $\mathcal{P} \cong Ch(Z)_*^*$, where Z is the final precubical set.

The final precubical set

Theorem [Paliga-Z, 2022]

Let Z be a final precubical set. Then

$$|\mathcal{P}| \cong \vec{P}(Z)^*_* \cong \coprod_{n\geq 0} \vec{P}(Z;n)^*_* \cong \coprod_{n\geq 0} Conf(n,\mathbb{R}^2).$$

As a consequence, $\vec{P}(Z;n)^*_* = K(B_n,1)$ (B_n denotes the braid group on n strands).

Applications

Every precubical set K has a unique (bipointed) precubical map $K \to Z$, which induces:

- a representation $\pi_1(\vec{P}(K;n)^w_v) \to B_n$,
- "characteric classes" in $H^*(\vec{P}(K;n)^w_v)$ induced by elements of $H^*(B_n)$.

What these invariants measure?

Towards concurrency (with U. Fahrenberg, C. Johansen, G. Struth)

Definition

- A *transition system* is a directed graph with edges labeled with letters of an *alphabet* Σ.
- An *automaton* is a transition system with distinguished "*start*" and "*accept*" vertices.
- Automata recognize *languages*: sets of words given by paths from start to accept states.
- Letters ("events") of words are totally ordered: no two events cannot be active simultaneously.

Definition (Pratt-van Glabbeek)

A *higher dimensional automaton* is a precubical set X with

- a *labeling* $\lambda : X[1] \to \Sigma$,
- *start states* $X_\perp \subseteq X[0]$,
- *accept states* $X^\top \subseteq X[0]$,
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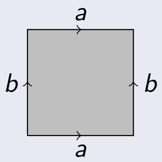

Looking for better definitions

Definition

- A *presheaf* over a category $\mathcal{C}$ is a contravariant functor $F : \mathcal{C}^{op} \to \mathbf{Set}$.
- An *element category* $El(F)$ of a presheaf F:
  - objects = pairs (c, x) such that $c \in \mathcal{C}$, $x \in F(x)$.
  - morphisms $(c, x) \to (c', x') = \{\alpha \in \mathcal{C}(x, x') \mid F(\alpha)(x') = x\}$.
- The canonical projection: $El(F) \ni (c, x) \mapsto c \in \mathcal{C}$.

Example

Directed graphs are presheaves over

Example

Transition systems are presheaves over $\mathcal{G}_\Sigma$,

- $Ob(\mathcal{G}_\Sigma) = \Sigma \cup \{\emptyset\}$,
- $\mathcal{G}_\Sigma(\emptyset, a) = \{d^0_a, d^1_a\}$ $(a \in \Sigma)$
- $\mathcal{G}_\Sigma(a, a) = \{id_a\}$, $\mathcal{G}_\Sigma(\emptyset, \emptyset) = \{id_\emptyset\}$
- no other morphisms

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$$\mathcal{G} = \quad \emptyset \underset{d^1}{\overset{d^0}{\rightrightarrows}} 1$$

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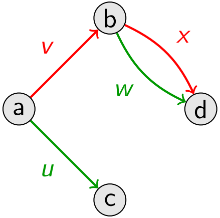
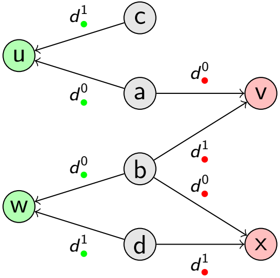
Example

$\Sigma = \{$●, ●$\}$

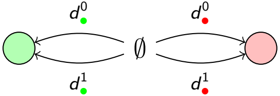
Transition system G



$El(G)$



$\mathcal{G}_\Sigma$



Lo-sets and lo-maps

Orders

We use two **strict** transitive relations: $<$ and $\dashrightarrow$.

- $p < q$ means that "p happens before q" (precedence),
- $p \dashrightarrow q$ means that "p has smaller id than q" (event order).

Definition

- An *lo-set* is a triple $U = (U, \dashrightarrow, \lambda)$, where
  - U is a finite set,
  - $\dashrightarrow$ is a (strict) total order on U,
  - $\lambda : U \to \Sigma$ is a labeling.
- An *lo-map* is an order- and label-preserving map $f : U \to V$ (it is always injective).
- Every lo-map $U \to V$ has the form ($A \subseteq V$)

$$\partial_A : U \cong V \setminus A \subseteq V.$$

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  - $\lambda : U \to \Sigma$ is a labeling.
- An *lo-map* is an order- and label-preserving map $f : U \to V$ (it is always injective).
- Every lo-map $U \to V$ has the form $(A \subseteq V)$

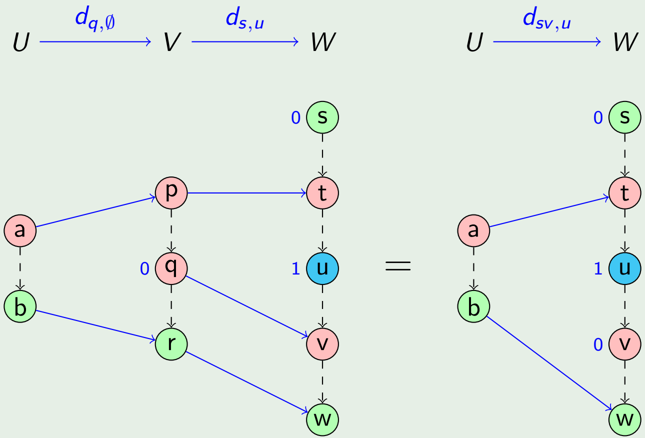
$$\partial_A : U \cong V \setminus A \subseteq V.$$

Precube maps

Definition

- A *precube map* from U to V is a triple (f, A, B), where $f : U \to V$ is an lo-map and
$$V = f(U) \dot{\cup} A \dot{\cup} B$$
- Every precube map has the form $(A, B \subseteq V,\ A \cap B = \emptyset)$
$$d_{A,B} = (\partial_{A\cup B}, A, B) : U \to V.$$
- Composition of precube maps $d_{A,B} : U \to V$ and $d_{C,D} : V \to W$
$$d_{C,D} \circ d_{A,B} = d_{\partial_{A\cup B}(A)\cup C, \partial_{A\cup B}(B)\cup D}.$$
- Notation: $d^0_A = d_{A,\emptyset}$, $d^1_B = d_{\emptyset,B}$.

Example (composition of precube maps)



Precubical sets

Definition

- The *precube category* $\square$ has lo-sets as objects and precube maps as morphisms.
- We do not distinguish the precube category and its skeleton (or the quotient by isomorphisms).
- Morphisms $d^0_A := d_{A,\emptyset} : U \cong V \setminus A \subseteq V$ are *forth-morphisms*.
- Morphisms $d^1_B := d_{\emptyset,B} : U \cong V \setminus B \subseteq V$ are *back-morphisms*.

Definition

A *precubical set* X is a presheaf over $\square$, ie, a functor $X : \square^{op} \to \mathbf{Set}$. Namely:

- For every $U = (a_1 \dashrightarrow \cdots \dashrightarrow a_n) \in \square$ there is a set $X[U]$.
- For $A, B \subseteq U \in \square$, $A \cap B = \emptyset$, there is a map

$$\delta_{A,B} = X[d_{A,B}] : X[U] \to X[U \setminus (A \cup B)].$$

- $\delta_{A,B} \circ \delta_{C,D} = \delta_{A\cup C, B\cup D} : X[U] \to X[U \setminus (A \dot\cup B \dot\cup C \dot\cup D)].$

Face maps

Let

- X be a precubical set ($X \in \square\mathbf{Set}$),
- $U = (u_1 \dashrightarrow u_2 \dashrightarrow \ldots \dashrightarrow u_n) \in \square$,
- $x \in X[U]$.

Geometry

- x is a cube with "directions" $u_1, \ldots, u_n$.
- $\delta^1_{u_k}(x)$ is the upper face of x in direction u_k.
- $\delta^0_{u_k}(x)$ is the lower face of x in direction u_k.
- $\delta_{A,B}(x)$ is an iterated face of x: lower in directions $a \in A$ and upper in directions $b \in B$.

Concurrency

- x is a state with active events $u_1, \ldots, u_n$.
- $\delta^1_{u_k}(x)$ is the state after terminating u_k.
- $\delta^0_{u_k}(x)$ is the state before starting u_k.
- $\delta_{A,B}(x)$ is the state obtained from x after terminating events $a \in A$ and "unstarting" events $b \in B$.

Higher dimensional automata

Definition

- The *cell category* **Cell**(X) of a precubical set X is its category of elements.
- **ev** : **Cell**$(X) \to \square$ is the canonical projection.

Definition

A *higher dimensional automaton* (HDA) is a precubical set X with

- the set $X_\perp \subseteq$ **Cell**(X) of *start cells*,
- the set $X^\top \subseteq$ **Cell**(X) of *accept cells*.

A HDA is *simple* if it has one start and one accept cell.
Precubical sets are regarded as HDA with no start/accept cells.

Definition

The *standard U-cube* $\square^U$ (for $U \in \square$) is the presheaf represented by U: $\square^U[V] = \square(V, U)$, with $(\square^U)_\perp = \{d_U^0 \in \square(\emptyset, U)\}$, $(\square^U)^\top = \{d_U^1 \in \square(\emptyset, U)\}$.

Higher dimensional automata

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Paths on HDA

Definition

A *path* in a HDA X is a sequence

$$\alpha = (x_0, \varphi_1, x_1, \varphi_2, \ldots, \varphi_n, x_n)$$

such that $x_k \in \mathbf{Cell}(X)$ and either

- $\varphi_k = \delta^0_A$ and $\delta^A_0(x_k) = x_{k-1}$ for $A \subseteq \mathbf{ev}(x_k)$ (*up-step*, notation: $x_{k-1} \nearrow^A x_k$) or
- $\varphi_k = \delta^1_B$ and $\delta^B_1(x_{k-1}) = x_k$ for $B \subseteq \mathbf{ev}(x_{k-1})$ (*down-step*, notation: $x_{k-1} \searrow_B x_k$).

Definition

Equivalence of paths $\alpha, \beta \in \mathrm{P}(X)$ ($\alpha \sim \beta$) is the equivalence relation spanned by

- $(x \nearrow^A y \nearrow^C z) \sim (x \nearrow^{A \cup C} z)$
- $(x \searrow^B y \searrow^D z) \sim (x \searrow^{B \cup D} z)$
- $\alpha \sim \beta \implies \gamma * \alpha * \delta \sim \gamma * \beta * \delta$.

Definition

Subsumption of paths $\alpha, \beta \in \mathrm{P}(X)$ ($\alpha \sqsubseteq \beta$) is the transitive relation spanned by

- $(y \searrow^B w \nearrow^A z) \sqsubseteq (y \nearrow^A x \searrow^B z)$
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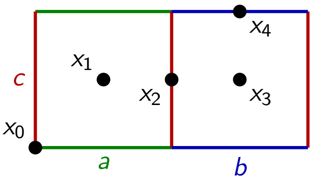
Paths: example

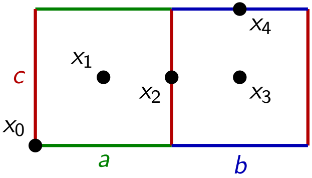


$$\alpha = (x_0 \nearrow^{ac} x_1 \searrow_a x_2 \nearrow^{b} x_3 \searrow_c x_4)$$

$$\beta = (x_0 \nearrow^{a} y \nearrow^{c} x_1 \searrow_a x_2 \nearrow^{b} x_3 \searrow_c x_4) \sim \alpha$$

$$\gamma = (x_0 \nearrow^{ac} x_1 \searrow_a x_2 \searrow^{c} z \nearrow_b x_4) \sqsubseteq \alpha$$

Paths: example



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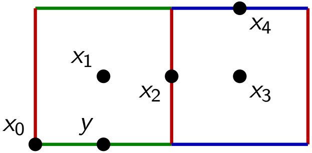

$$\beta = (x_0 \nearrow^{a} y \nearrow^{c} x_1 \searrow_a x_2 \nearrow^b x_3 \searrow_c x_4) \sim \alpha$$

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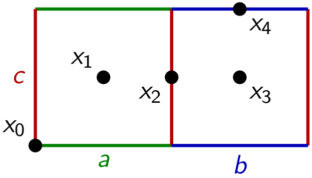

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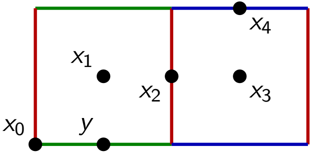
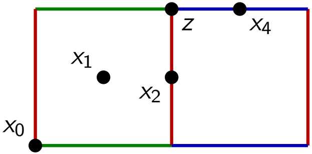


$$\beta = (x_0 \nearrow^a y \nearrow^c x_1 \searrow_a x_2 \nearrow^b x_3 \searrow_c x_4) \sim \alpha$$



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Paths as functors

Definition

Directed category is a category $\mathcal{C}$ with wide subcategories $\mathcal{C}_0 \subseteq \mathcal{C} \supseteq \mathcal{C}_1$.
Morphisms of $\mathcal{C}_0$ are *forth-morphisms*, morphisms of $\mathcal{C}_1$, *back-morphisms*.
A functor is *directed* if it preserves forth- and back-morphism.

Examples

- Category $\square$: d_A^0 are forth-morphisms, d_B^1 are back-morphisms.
- The category of cells $\mathbf{Cell}(X)$ is directed: a morphism $(x, U) \xrightarrow{\varphi} (y, V)$ is a forth/back-morphism if $\varphi \in \square(U, V)$ is such.
  Further, $\mathbf{ev} : \mathbf{Cell}(X) \to \square$ is a directed functor.
- *Linear categories* ($\longrightarrow$ are forth-morphisms, $\longleftarrow$ are back-morphisms)

$$\bot = 0 \longrightarrow 1 \longleftarrow 2 \longleftarrow 3 \longrightarrow 4 \longleftarrow 5 \longrightarrow 6 \longrightarrow 7 \longleftarrow \cdots \longrightarrow n = \top$$

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Labels of paths

Definition

A *path* on HDA X is a directed functor $\alpha : \mathcal{L} \to \mathbf{Cell}(X)$ from a linear category $\mathcal{L}$.

Definition

The *label* of a path $\alpha : \mathcal{L} \to \mathbf{Cell}(X)$ is a simple HDA

$$\lambda(\alpha) = colim\left(\mathcal{L} \xrightarrow{\alpha} \mathbf{Cell}(X) \xrightarrow{ev} \square \xrightarrow{\text{Yoneda}} \square\mathbf{Set}\right) \in \square\mathbf{Set}$$

with $\lambda(\alpha)_\perp = \alpha(\perp)$, $\lambda(\alpha)^\top = \alpha(\top)$.

Remark

Not every simple HDA may be a label of a path.

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Remark

Not every simple HDA may be a label of a path.

Tracks

Definition

- A *track object* is a simple HDA having the form

$$T = colim\left(\mathcal{L} \xrightarrow{\omega} \square \xrightarrow{\text{Yoneda}} \square\mathbf{Set}\right),$$

$$T_{\bot} = \omega(\bot_{\mathcal{L}}), \qquad T^{\top} = \omega(\top_{\mathcal{L}})$$

- A *track* in HDA X is a precubical map $\alpha : T \to X$ from a track object T.
- The *label* of a track α is T itself.

Proposition

There is a natural label-preserving bijection between

- Tracks on X.
- Equivalence classes of paths on X.

Subsumption of paths corresponds to inclusion of tracks.

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Subsumption of paths corresponds to inclusion of tracks.

The category of track objects

Definition

The 2-category of tracks objects **TrO**:

- Objects are lo-sets ($Ob(\mathbf{TrO}) = Ob(\square)$)
- Morphisms from U to V are (isomorphisms classes of) tracks objects T such that $\mathbf{ev}(T_\perp) = U$ and $\mathbf{ev}(T^\top) = V$.
- Composition of $T \in \mathbf{TrO}(U, V)$ and $T' \in \mathcal{T}(V, W)$ is

$$T * T' = colim \left(T \xleftarrow{\top} \square^V \xrightarrow{\perp} T' \right).$$

- 2-morphisms $T \Rightarrow T'$ are HDA-maps (subsumptions).
- 2-composition is the composition of HDA-maps.

The category of tracks

Definition

The *track complex* $\mathbf{Tr}(X)$ of a precubical set X is a 2-category:

- Objects are cells of X ($Ob(\mathbf{Tr}(X)) = Ob(\mathbf{Cell}(X))$)
- Morphisms from x to y are tracks $\boldsymbol{\alpha} : T \to X$ from x to y (ie, $\boldsymbol{\alpha}(T_\perp) = x$, $\boldsymbol{\alpha}(T^\top) = y$).
- Composition of $\boldsymbol{\alpha} : T \to X$ and $\boldsymbol{\beta} : T' \to X$ is the concatenation

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The forgetful functor $\mathbf{Tr}(X) \to \mathbf{TrO}$ is a “presheaf” on $\mathbf{TrO}$.

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Understanding track objects: ipomsets

Definition

An *ipomset* is a tuple $(P, \lambda, <, \dashrightarrow, S, T)$, where

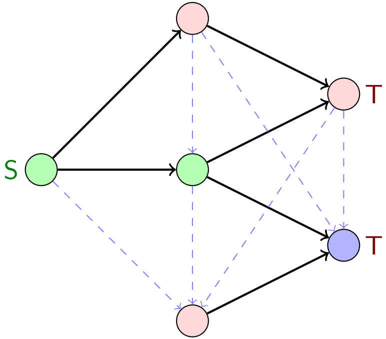
- P is a finite set,
- $\lambda : P \to \Sigma$ is a *labelling*,
- $<$ is a partial order on P (*precedence order*),
- $\dashrightarrow$ is a partial order on P (*event order*),
- $S \subseteq P$ is a subset of $<$-minimal elements of P (*source interface*),
- $T \subseteq P$ is a subset of $<$-maximal elements of P (*target interface*).

Elements $p, q \in P$ are *parallel* ($p \parallel q$) if $p \neq q$, $p \not< q$ and $q \not< p$.
We require that

- If $p \parallel q$, then $p \dashrightarrow q$ or $q \dashrightarrow p$.

An ipomset is *interval* if $(P, <)$ is an interval order.

Ipomsets: an example



colors = labels

⟶ precedence

⇢ event order

S source interface

T target interface

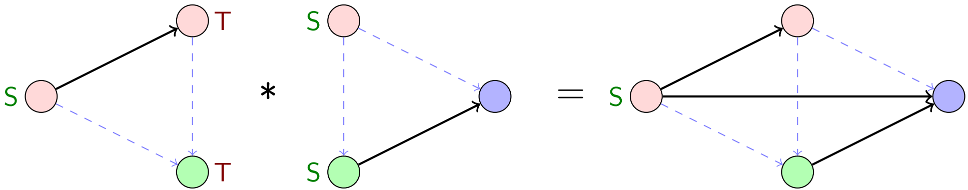
Serial composition of ipomsets

Definition

A *serial composition* of ipomsets P, Q such that $T_P \simeq S_Q$ is

$$P * Q = (P \,\dot\cup\, Q)/\, T_P \sim S_Q$$

- $r <_{P*Q} s$ if $r <_P s$ or $r <_Q s$ or $r \in P \setminus T_P, s \in Q \setminus S_Q$,
- $\dashrightarrow_{P*Q}$ is the transitive closure of $<_P \cup <_Q$,
- $S_{P*Q} = S_P$, $T_{P*Q} = T_Q$.

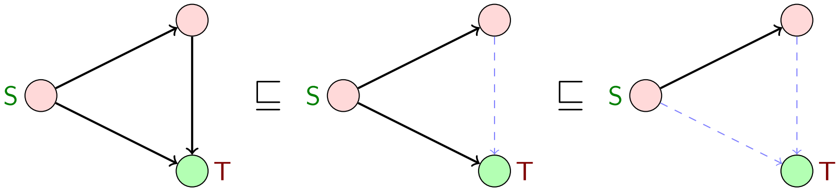


Subsumption of ipomsets

Definition

A *subsumption* of ipomsets ($P \sqsubseteq Q$) is a bijective map $f : P \to Q$ that

- preserves labels ($\lambda(f(p)) = \lambda(p)$),
- reflects precedence ($f(p) < f(p') \implies p < p'$),
- preserves essential event order ($p \parallel p' \wedge p \dashrightarrow p' \implies f(p) \dashrightarrow f(p')$),
- preserves interfaces ($f(S_P) = S_Q$, $f(T_P) = T_Q$).



Ipomset category

Definition

The 2-category of ipomsets **iPoms**:

- Objects are lo-sets ($Ob(\mathbf{iPoms}) = Ob(\square)$)
- Morphisms from U to V are (isomorphisms classes of) ipomsets P such that $S_P \cong U$ and $T_P \cong V$.
- Composition of $P \in \mathbf{iPoms}(U, V)$ and $Q \in \mathbf{iPoms}(V, W)$ is

$$P * Q \in \mathbf{iPoms}(U, W).$$

- 2-morphisms $P \Rightarrow Q$ are subsumptions $f : P \sqsubseteq Q$.
- 2-composition is the composition of subsumptions.

Let **iiPoms** $\subseteq$ **iPoms** be the full subcategory of interval ipomsets.

Theorem

There is a natural 2-equivalence $\mathbf{iiPoms} \ni P \mapsto \square^P \in \mathbf{TrO}$.

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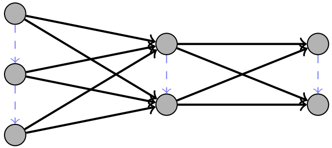
Conclusions

- The track complex $\mathbf{Tr}(X)$ admits a functor $\mathbf{ev} : \mathbf{Tr}(X) \to \mathbf{iiPoms}$ that makes it a "presheaf" over **iiPoms**.
- The cube chain category $\mathcal{P}$ is a full subcategory of **iiPoms** consisting of serial compositions of discrete ipomsets:



- "Taming" theorem for track complexes: every track complex is determined uniquely by its values on $\mathcal{P} \subseteq \mathbf{iiPoms}$ (it is a "sheaf").

Appendix: languages of HDA and Kleene theorem

Definition

Let X be a HDA.

- A track $\alpha : T \to X$ is *accepting* if $\alpha(T_\perp) \in X_\perp$ and $\alpha(T^\top) \in X^\top$.
- The *language* of X is $\mathbf{Lang}(X) = \{P \in \mathbf{iiPoms} \mid \mathbf{HDA}(\square^P, X) \neq \emptyset\}$.

Definition

A language $L \subseteq \mathbf{iiPoms}$ is *regular* if $L = \mathbf{Lang}(X)$ for a finite HDA X.

Kleene theorem for HDA

The family of regular languages is the concurrent Kleene algebra generated from singleton languages by unions, serial compositions, parallel compositions and Kleene plus.