From Discrete Morse Theory to Combinatorial Topological Dynamics

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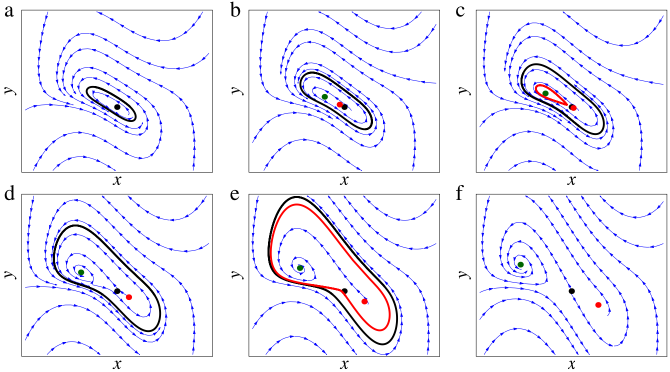
Topological Analysis of Dynamical Systems

Differential equations of the form $\dot{x} = f(x)$ generate continuous-time dynamical systems $\varphi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ under suitable conditions. Trying to understand their dynamics naturally leads to the study of bounded invariant sets, which in turn often relies on topological methods.



[Figure from Engler et al., *Dynamical systems analysis of the Maasch-Saltzman model for glacial cycles*, Physica D (2017).]

Invariance in Dynamical Systems

An invariant set is a subset $S \subset \mathbb{R}^d$ of phase space which satisfies

$$\varphi(t, x) \in S \quad \text{for all} \quad t \in \mathbb{R} \quad \text{and} \quad x \in S \, .$$

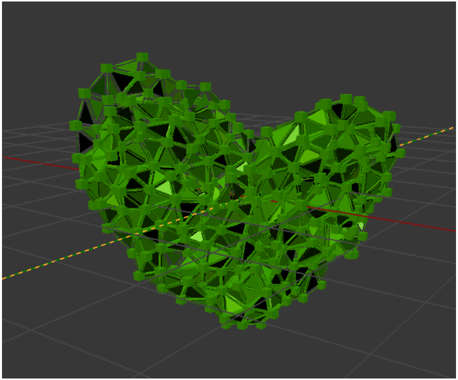
Their study is part of a qualitative theory, which can provide an overall description of the structure of the dynamics of a given model without actually solving the underlying equation. It involves the following steps:

- Equilibrium Solutions: Locate solutions of the differential equation which are constant in time.
- Periodic Solutions: These are solutions $u(t)$ with $u(t + T) = u(t)$ for all $t \in \mathbb{R}$, for some period $T > 0$.
- Global Dynamics: How can we make transitions between specific solutions? For example, are there solutions which converge to equilibria in forward and backward time?

Topological methods can be used to study invariant sets!

Sampled Dynamics and Topological Methods

Topological methods have been used successfully for rigorous numerics and computer-assisted proofs in dynamics. More recently, they have opened the way to applications for sampled dynamics, based on combinatorial multivector fields in finite topological spaces.



Combinatorial multivector field around the Lorenz attractor constructed from a vector field sample [Mateusz Juda (2018)]

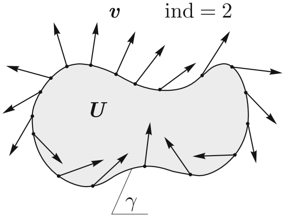
Outline

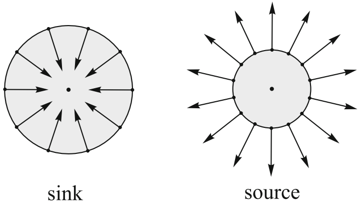
Locating Equilibrium Solutions via Topology

Locating equilibria of flows might be difficult:

- Nonlinear systems of equations can rarely be solved explicitly.
- Numerical results might not be enough, or spurious.

Topological methods such as the Brouwer degree and the associated winding number can be used to establish the existence of equilibria by studying the vector field far from the actual solution. This leads to results that are robust under small perturbations.





[Figures from Bolsinov et al., *Bifurcation analysis and the Conley index in mechanics*, Regular and Chaotic Dynamics (2012).]

Isolated Invariant Sets

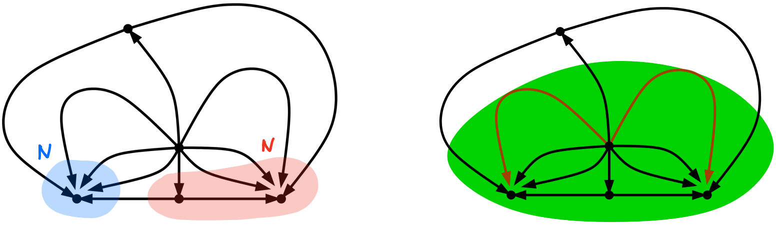
Conley, Easton (1971):

In order to study more elaborate perturbation insensitive invariance and stability one has to restrict attention to isolated invariant sets:

- A compact set N is an isolating neighborhood if

$$\operatorname{Inv}(N, \varphi) := \{x \in N \ : \ \varphi(\mathbb{R}, x) \subset N\} \ \subset \ \operatorname{int} N$$

- A set S is an isolated invariant set if $S = \operatorname{Inv}(N, \varphi)$ for some isolating neighborhood N.



Conley Index of Isolated Invariant Sets

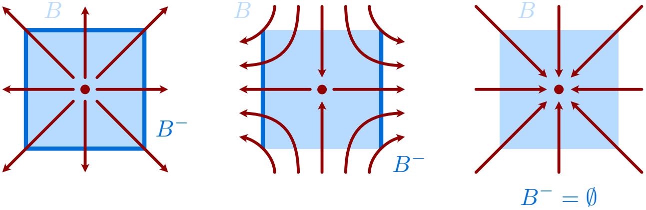
Conley (1978): Degree theory for isolated invariant sets

One definition of the Conley index uses special isolating neighborhoods:

- A compact set B is called an isolating block, if its exit set

$$B^- := \{x \in B \ : \ \varphi([0,T),x) \not\subset B \ \text{ for all } \ T > 0\}$$

  is closed, and if there are no internal flow tangencies at the boundary ∂B.
- Every isolated invariant set has associated isolating blocks.

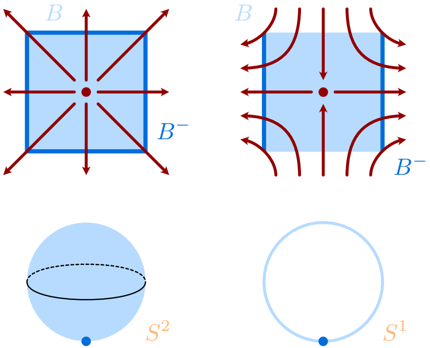


Conley Index of Isolated Invariant Sets

If S is an isolated invariant set and B is an isolating block for S, then the Conley index of S is defined as the homotopy type or homology of the pointed topological space $(B/B^-, [B^-])$. The homological Conley index is

$$CH_*(S) = H_*(B/B^-, [B^-]) \approx H_*(B, B^-)$$

For hyperbolic equilibria the homotopy indices are pointed spheres.



Properties of the Conley Index

Isolating blocks are rather restrictive, and one can introduce the more general concept of index pairs to compute the Conley index.

Important properties:

- The Conley index is well-defined and only depends on the isolated invariant set S. Nevertheless, it can be computed from the pair (B, B^-) even if S itself is unknown.
- Ważewski Property: If the Conley index is not trivial, then necessarily one has $S \neq \emptyset$.
- By adding additional information, the Conley index can be used to establish more detailed information, such as the existence of periodic orbits, heteroclinic orbits, and even chaos.

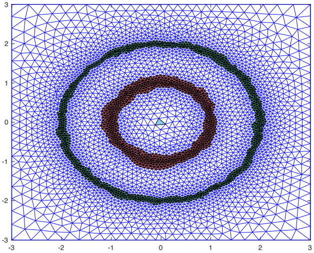
Conley index was developed as a generalization of the Morse index of hyperbolic equilibrium solutions. It allows for the generalization of Morse theory to hierarchies of isolated invariant sets called Morse decompositions.

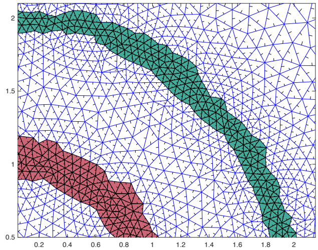
Computer-Assisted Proofs via Discretizations

Many computational techniques have been developed to find isolating blocks or index pairs for dynamical systems generated by maps, multi-valued maps, and flows, based on triangulations and other discretizations. Combined with interval arithmetic, they can lead to computer-assisted existence proofs for specific dynamical structures.

Mrozek, Srzednicki, Thorpe, W. (2022):

Existence of periodic orbits in arbitrary dimensions via flow-transverse phase space decompositions.



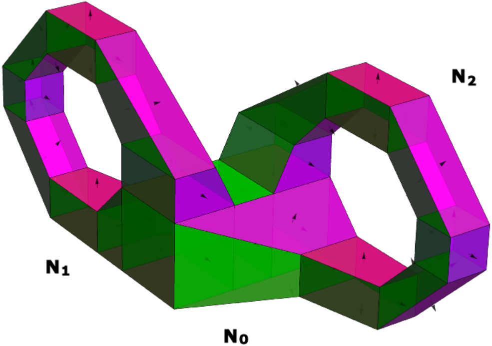


Chaos via Phase Space Decompositions

These techniques have the potential to algorithmically prove chaos in differential equations by identifying combinatorial flow structures that imply complicated dynamical behavior.

Mrozek, Srzednicki, Thorpe, W. (2022):

Lorenz-like structure with transverse flow panels along the boundary which automatically yields infinitely many periodic orbits in its interior.



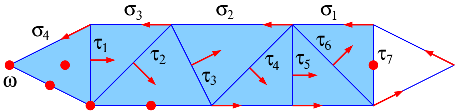
Outline

1. Topological Analysis of Dynamical Systems
   - Isolated Invariant Sets
   - Isolating Blocks & Conley Index
   - Discretization & Computer-Assisted Proofs
2. Forman's Combinatorial Vector Fields
   - Isolation in the Combinatorial Setting
   - Conley Index & Morse Decompositions
   - Associated Classical Dynamics
3. Combinatorial Topological Dynamics
   - Finite Topological Spaces
   - Dynamics through Multivectors
   - Conley-Morse Theory for Multivectors

Forman's Combinatorial Vector Fields

Forman's combinatorial vector fields have found numerous applications in areas such as visualization and mesh compression, astronomy, homology computations, and many others. But from a purely dynamical point of view, they raise the following questions:

- Is it possible to formulate a qualitative dynamical theory directly in the combinatorial setting, in the sense of Conley index and Morse decompositions?
- Are there formal links between combinatorial dynamics and classical dynamics? Does every combinatorial vector field give rise to a classical semiflow on the underlying polytope?



Combinatorial Vector Field:

Critical cells (index is given by the cell dimension)

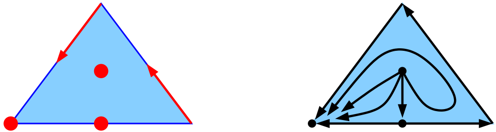
Arrows (from facet to simplex)

Dynamics Induced by a Combinatorial Vector Field

Let $\mathcal{V}$ denote a combinatorial vector field on a simplicial complex $\mathcal{X}$, that is, a partition of $\mathcal{X}$ into singletons $\{\sigma\}$ and doubletons $\{\sigma^-, \sigma^+\}$, where we always assume that $\dim \sigma^+ = 1 + \dim \sigma^-$.

In its original form, a combinatorial vector field $\mathcal{V}$ does not automatically define a dynamical system. However, our intuition implies the following:

- Critical cells σ should lead to both fixed points and flow towards the boundary of the respective simplex.
- Arrow sources σ^- should always lead to flow towards the arrow target simplex σ^+.
- Arrow targets σ^+ should always lead to flow towards the boundary of the simplex σ^+, but not towards the face σ^-.

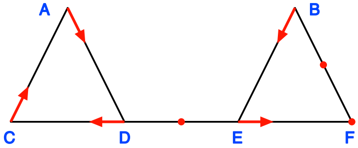


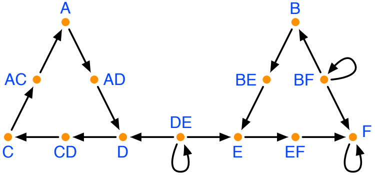
The Multivalued Flow-Map $\Pi_\mathcal{V}$

We can therefore think of the dynamics induced by a combinatorial vector field $\mathcal{V}$ on the simplicial complex $\mathcal{X}$ as the iteration of an associated multivalued flow-map $\Pi_\mathcal{V} : \mathcal{X} \multimap \mathcal{X}$ defined by

$$\Pi_\mathcal{V}(\tau) := \begin{cases} \operatorname{Cl}\sigma & \text{if } \tau = \sigma \text{ is a critical cell ,} \\ \{\sigma^+\} & \text{if } \tau = \sigma^- \text{ is an arrow source ,} \\ \operatorname{Bd}\sigma^+ \setminus \{\sigma^-\} & \text{if } \tau = \sigma^+ \text{ is an arrow target .} \end{cases}$$

Cl/Bd denote the combinatorial closure/boundary of a simplex.





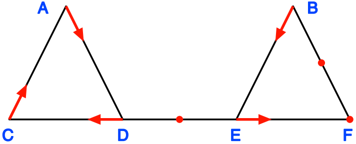
Orbits of a Combinatorial Vector Field $\mathcal{V}$

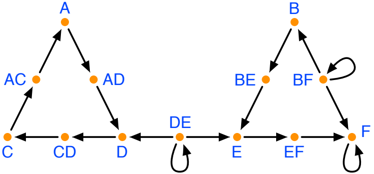
With these definitions, a solution ρ, or orbit, of the field $\mathcal{V}$ is a map $\rho : I \to \mathcal{X}$, where I denotes an interval in $\mathbb{Z}$, such that

$$\boxed{\rho_{k+1} \in \Pi_{\mathcal{V}}(\rho_k) \quad \text{for all} \quad k, k+1 \in I}$$

A full solution through $\sigma \in \mathcal{X}$ is a solution with $I = \mathbb{Z}$ and $\rho_0 = \sigma$.

For the combinatorial vector field $\mathcal{V}$ below, one can find three equilibria, one periodic orbit, as well as a number of heteroclinic orbits. All of these correspond to paths in the associated digraph.

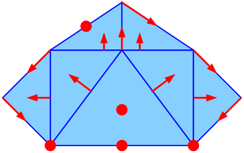


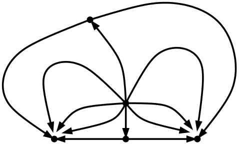


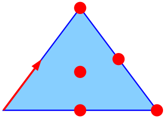
Invariant Sets for Combinatorial Vector Fields

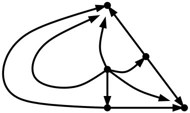
Let $\mathcal{V}$ denote a combinatorial vector field on a finite simplicial complex $\mathcal{X}$. Then a set $\mathcal{S} \subset \mathcal{X}$ is called an invariant set for the associated flow $\Pi_{\mathcal{V}}$, if for each simplex $\sigma \in \mathcal{S}$ there exists a full solution $\rho : \mathbb{Z} \to \mathcal{X}$ through σ which lies completely in $\mathcal{S}$.

The following two examples exhibit a wide variety of invariant sets.







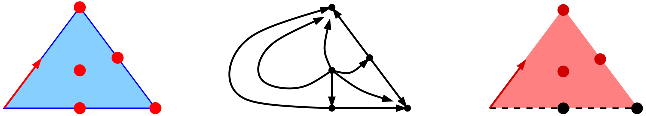


But what about isolation? This requires topology!

The Simplicial Complex as Finite Topological Space

The face relation on a finite simplicial complex $\mathcal{X}$ defines the Alexandrov topology, and turns $\mathcal{X}$ into a finite T_0 topological space. A subset $\mathcal{A} \subset \mathcal{X}$ is open in this topology if all cofaces of any element of $\mathcal{A}$ are also in $\mathcal{A}$. The closure of $\mathcal{A}$, denoted by $\operatorname{Cl} \mathcal{A}$, is the family of all faces of all simplices in $\mathcal{A}$.

Considering $\mathcal{X}$ as finite T_0 space has advantages. For example, singular homology groups are automatically defined. But we have to accept that interesting invariant sets might no longer be closed.

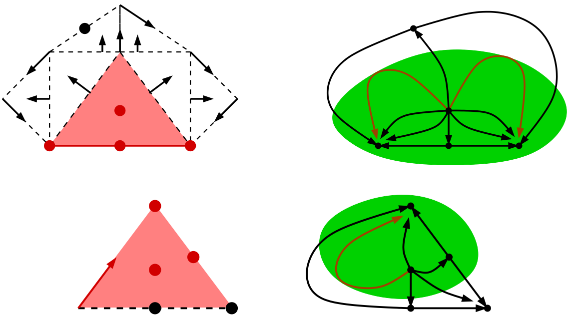


The set $\mathcal{S}$ which contains the top vertex, the two top edges, and the 2-simplex, is invariant, but not closed. It is in fact open.

Non-Isolated Invariant Sets

General invariant sets are difficult to study. Therefore, Conley proposed to focus on isolated invariant sets. In the classical theory, these are invariant sets S for which there exists a compact set N such that S is the largest invariant set in N, as well as $S \subset \operatorname{int} N$.

Using classical flows associated with $\mathcal{V}$, one can see that also in the combinatorial setting there should be a notion of isolation.



Isolated Invariant Set

Let $\mathcal{S} \subset \mathcal{X}$ denote an invariant set for $\Pi_\mathcal{V}$, and define the exit set or mouth of $\mathcal{S}$ by

$$\boxed{\operatorname{Mo}\mathcal{S} := \operatorname{Cl}\mathcal{S} \setminus \mathcal{S}\ .}$$

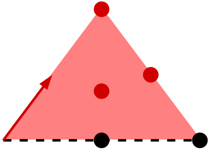
Then the invariant set $\mathcal{S}$ is an isolated invariant set, if we have:

(a) The exit set $\operatorname{Mo}\mathcal{S}$ is closed in the simplicial complex $\mathcal{X}$.

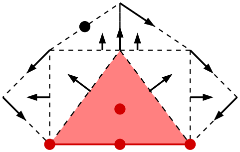
(b) There exists no solution $\rho : [-1, 1] \cap \mathbb{Z} \to \mathcal{X}$ of $\Pi_\mathcal{V}$ such that both $\rho_{-1} \in \mathcal{S}$ and $\rho_1 \in \mathcal{S}$ hold, as well as $\rho_0 \in \operatorname{Mo}\mathcal{S}$.

The closure $\operatorname{Cl}\mathcal{S}$ is called an isolating block for the isolated invariant set $\mathcal{S}$. Note that (b) rules out internal tangencies, analogous to the classical case.

(a), not (b):



(b), not (a):



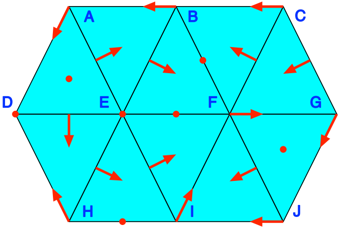
Isolated Invariant Set

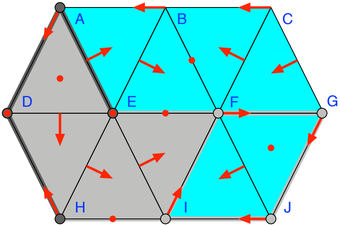
It is possible to characterize isolated invariant sets purely through the combinatorial vector field $\mathcal{V}$:

Theorem (Kaczynski, Mrozek, W., 2016)

Let $\mathcal{S} \subset \mathcal{X}$ be an invariant set for $\Pi_{\mathcal{V}}$. Then $\mathcal{S}$ is an isolated invariant set if and only if $\mathrm{Mo}\,\mathcal{S}$ is closed, and every arrow of $\mathcal{V}$ either lies completely in $\mathcal{S}$ or completely outside of $\mathcal{S}$.

In the example, the invariant set $\mathcal{S}$ is shown in light gray, while its exit set $\mathrm{Mo}\,\mathcal{S}$ is dark gray.





Conley Index of an Isolated Invariant Set

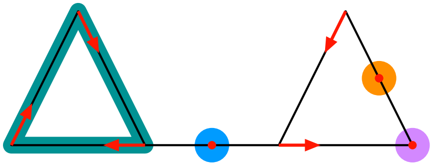
Definition (Conley Index)

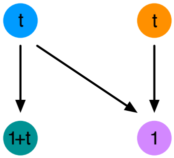
Let $\mathcal{S} \subset \mathcal{X}$ be an isolated invariant set for $\Pi_{\mathcal{V}}$. Then the Conley index of $\mathcal{S}$ is the (simplicial) homology

$$C_*(\mathcal{S}) := H_*(\mathrm{Cl}\,\mathcal{S}, \mathrm{Mo}\,\mathcal{S}) \ .$$

The Poincaré polynomial of $\mathcal{S}$ is defined by

$$p_{\mathcal{S}}(t) := \sum_{k=0}^{\infty} \beta_k(\mathcal{S}) t^k \ , \quad \text{where} \quad \beta_k(\mathcal{S}) = \mathrm{rank}\, C_k(\mathcal{S}) \ .$$





Morse Decompositions and Conley-Morse Graphs

Morse decompositions can be defined using the notion of limit sets:

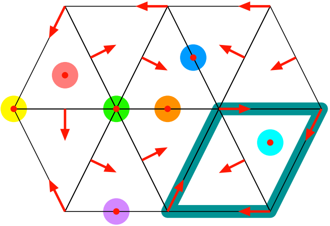
- For a solution $\rho : [a, \infty) \cap \mathbb{Z} \to \mathcal{X}$, we define the ω-limit set as the intersection $\omega(\rho) = \cap_{n \geq a} \{\rho_k : k \geq n\}$. Similarly for α-limit sets.

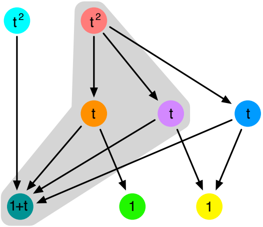
A family $\mathcal{M} = \{\mathcal{M}_p \,|\, p \in \mathbb{P}\}$, where $\mathbb{P}$ is a poset, of disjoint isolated invariant subsets of $\mathcal{X}$ is a Morse decomposition of $\mathcal{X}$, if the following hold:

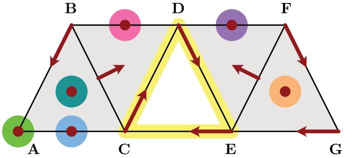
- For every solution ρ we have $\alpha(\rho) \subset \mathcal{M}_p$ and $\omega(\rho) \subset \mathcal{M}_q$ for some $p \geq q$, as long as the limit sets are defined. If in addition $p = q$, then we require $\operatorname{im} \rho \subset \mathcal{M}_p$.
- The associated Conley-Morse graph is the partial order induced on $\mathcal{M}$ by the existence of connections, and represented as a directed graph labelled with the Conley indices of the isolated invariant sets in $\mathcal{M}$ in terms of their Poincaré polynomials.

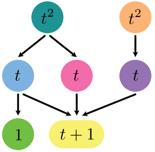
Morse Decompositions and Conley-Morse Graphs

Given a combinatorial vector field $\mathcal{V}$ on a simplicial complex $\mathcal{X}$, one can easily find the finest Morse decomposition $\mathcal{M}$ by determining the strongly connected path components of the digraph associated with the multivalued map $\Pi_{\mathcal{V}} : \mathcal{X} \multimap \mathcal{X}$.









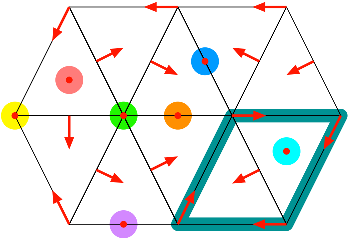
Morse Inequalities for Morse Decompositions

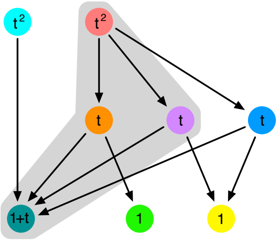
It is possible to obtain analogues of the classical Morse inequalities:

Theorem (Mrozek, 2017)

Let $\mathcal{S} \subset \mathcal{X}$ be an invariant set for $\Pi_{\mathcal{V}}$, and let $\mathcal{M} = \{\mathcal{M}_a \mid a \in \mathbb{P}\}$ be a Morse decomposition of $\mathcal{S}$. Then for a polynomial q with nonnegative coefficients one has

$$\sum_{a \in \mathbb{P}} p_{\mathcal{M}_a}(t) = p_{\mathcal{S}}(t) + (1+t)q(t) \; .$$



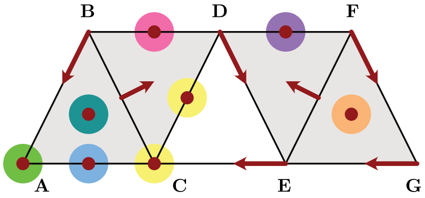


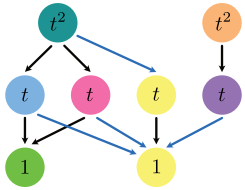
$$2t^2 + 4t + 3 = 1 + 2(1+t)^2$$

Connection Matrices

Mrozek, W. (2022):

- Connection matrices can be defined directly for $\mathcal{V}$, and they are the analogue of the boundary operator in the Morse complex.
- Nonzero entries in a connection matrix guarantee connecting orbits between the associated Morse sets, just as in the Morse complex.
- These connection matrices can be computed explicitly using the algorithms of Harker, Mischaikow, Spendlove (2018) and Dey, Lipinski, Mrozek, Slechta (2022).

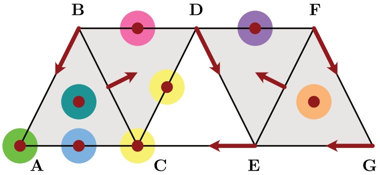


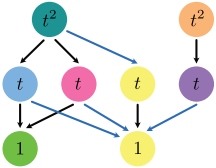


Uniqueness of Connection Matrices

Theorem (Mrozek, W., 2022)

If $\mathcal{V}$ is a gradient combinatorial vector field and $\mathcal{M}$ its finest Morse decomposition, then the connection matrix is uniquely determined.





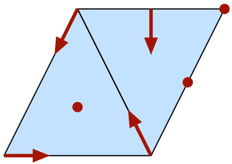
	CD	**AC**	**BD**	**DF**	**ABC**	**EFG**
A	0	1	1	0		
C	0	1	1	0		
CD					1	0
AC					1	0
BD					1	0
DF					0	1

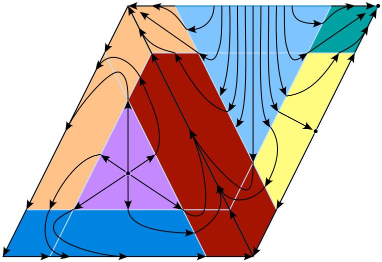
Linking Combinatorial Dynamics and Classical Semiflows

The dynamics of a combinatorial vector field $\mathcal{V}$ on a simplicial complex $\mathcal{X}$ can always be represented as a classical semiflow.

Theorem (Mrozek, W., 2021)

For every combinatorial vector field $\mathcal{V}$ on a simplicial complex $\mathcal{X}$ one can construct a classical semiflow $\varphi : \mathbb{R}_0^+ \times X \to X$ on any geometric realization X of $\mathcal{X}$ which exhibits the same dynamics as $\mathcal{V}$ in the sense of Conley-Morse graphs.



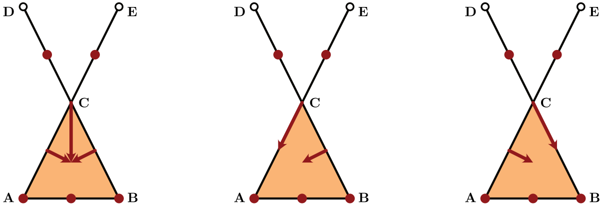


Outline

Constraints Imposed by Forman Vector Fields

While the Conley-Morse theory for combinatorial vector fields successfully mimics its classical counterpart, Forman vector fields on simplicial complexes often prove to be too restrictive:

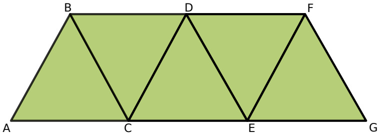
- Constructing flow transverse phase space decompositions from classical vector fields becomes more feasible if one allows for polygonal or even more complicated regions.
- In applications based on discrete data, precise flow directions might not be known, and flexibility is needed to capture the actual dynamical possibilities.

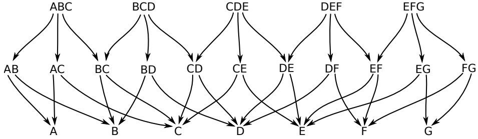


Extension to Finite Topological Spaces

Constraints imposed by the use of simplicial complexes can be removed by considering combinatorial dynamics on finite topological spaces.

- Finite T_0 topological spaces X correspond to partially ordered sets.
- For every point $x \in X$ there exists a smallest closed set $\operatorname{cl} x$ which contains x.
- The partial order on X is defined as $x \leq y$ if and only if $x \in \operatorname{cl} y$.
- Finite T_0 topological spaces generalize simplicial complexes, cellular complexes, as well as Lefschetz complexes.



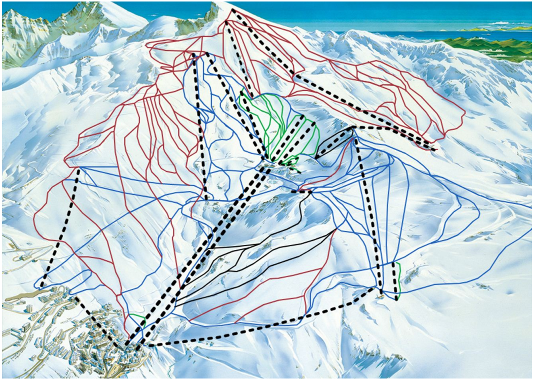


Dynamics through Multivectors

- Motivated by the case of combinatorial vector fields, the default dynamical behavior wants to move points downwards in the poset. The inherent flow is towards the boundary.
- This is analogous to pure gradient dynamics.

Dynamics through Multivectors

- In order to obtain systems with interesting dynamics we introduce lifts, i.e., multivectors which allow one to move upwards.
- A multivector is any convex set in the poset. Equivalently, such sets are locally closed. This generalizes Forman's arrows and critical cells.



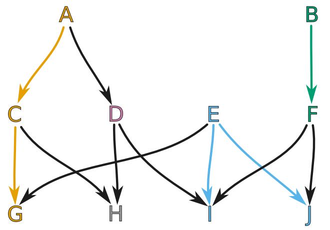
Multivector Fields on Finite Topological Spaces

Definition (Multivector Field)

A multivector field $\mathcal{V}$ on a finite topological space X is a partition of X into locally closed sets.

For each multivector $V \in \mathcal{V}$ the relative homology $H_*(\mathrm{cl}\, V, \mathrm{mo}\, V)$ is well-defined, and it allows for the following classification:

- Critical multivector: $H_*(\mathrm{cl}\, V, \mathrm{mo}\, V) \neq 0$ (e.g. Forman's singleton)
- Regular multivector: $H_*(\mathrm{cl}\, V, \mathrm{mo}\, V) = 0$ (e.g. Forman's doubleton)



$\mathcal{V} = \{\{A, C, G\}, \{D\}, \{H\}, \{E, I, J\}, \{B, F\}\}$

Critical: $\{D\}$, $\{H\}$, and $\{B, F\}$

Regular: $\{A, C, G\}$ and $\{E, I, J\}$

Combinatorial Flow for a Multivector Field

We define the combinatorial flow associated with the multivector field $\mathcal{V}$ as the multivalued map $\Pi_{\mathcal{V}} : X \multimap X$ given by

$$\boxed{\Pi_{\mathcal{V}}(x) := \operatorname{cl} x \cup [x]_{\mathcal{V}}}$$

where $[x]_{\mathcal{V}}$ denotes the unique multivector in $\mathcal{V}$ containing x.

- The flow towards the boundary is encapsulated in the $\operatorname{cl} x$ part of the image. The lift motion is encoded in $[x]_{\mathcal{V}}$.
- Solutions $\rho : \mathbb{Z} \to X$ can be defined as in the Forman case, but this would imply that every subset of X is invariant.
- We therefore only consider essential solutions. They are characterized by the property that if $\rho(k)$ lies in a regular multivector V, then there are $\ell_1 < k < \ell_2$ with $\rho(\ell_i) \notin V$.

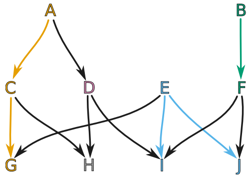
Essential solutions can only remain in critical multivectors for infinite time, they have to exit regular multivectors in finite forward and backward time.

Conley-Morse Theory for Multivector Fields

Theorem (Kubica, Lipinski, Mrozek, W., 2019)

Based on earlier work by Mrozek (2017), one can develop a complete Conley-Morse theory for combinatorial multivector fields on finite T_0 topological spaces. This leads to notions of isolated invariant sets, Conley index, Morse decompositions, and the Morse inequalities.

- Invariant sets are isolated invariant sets if they are locally closed and $\mathcal{V}$-compatible. We say that a set is $\mathcal{V}$-compatible, if it is the union of a collection of multivectors.
- The homological Conley index of an isolated invariant set S is given by the relative homology $C_*(S) = H_*(\mathrm{cl}\, S, \mathrm{mo}\, S)$.



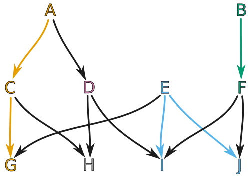
$\mathcal{M}_3 = \{B, F\}$	$C_*(\mathcal{M}_3) = \tilde{H}_*(S^1)$
$\mathcal{M}_2 = \{A, C, D, E, G, I, J\}$	$C_*(\mathcal{M}_2) = \tilde{H}_*(S^1)$
$\mathcal{M}_1 = \{H\}$	$C_*(\mathcal{M}_1) = \tilde{H}_*(S^0)$

Conley-Morse Theory for Multivector Fields

Theorem (Kubica, Lipinski, Mrozek, W., 2019)

Based on earlier work by Mrozek (2017), one can develop a complete Conley-Morse theory for combinatorial multivector fields on finite T_0 topological spaces. This leads to notions of isolated invariant sets, Conley index, Morse decompositions, and the Morse inequalities.

- Finest Morse decompositions can be found via strongly connected components of the digraph associated with $\Pi_{\mathcal{V}}$ which contain essential solutions.
- Attractors are precisely given by closed isolated invariant sets, while repellers are open isolated invariant sets.



$$\mathcal{M}_3 = \{B, F\} \qquad C_*(\mathcal{M}_3) = \tilde{H}_*(S^1)$$

$$\mathcal{M}_2 = \{A, C, D, E, G, I, J\} \qquad C_*(\mathcal{M}_2) = \tilde{H}_*(S^1)$$

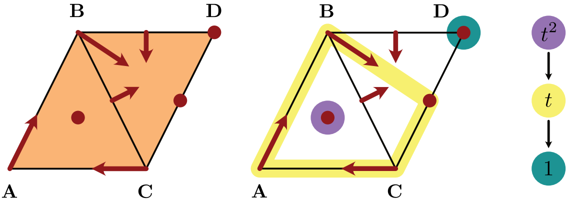
$$\mathcal{M}_1 = \{H\} \qquad C_*(\mathcal{M}_1) = \tilde{H}_*(S^0)$$

Connection Matrices for Multivector Fields

Theorem (Mrozek, W., 2022)

The existence of connection matrices can be established for multivector fields on Lefschetz complexes.

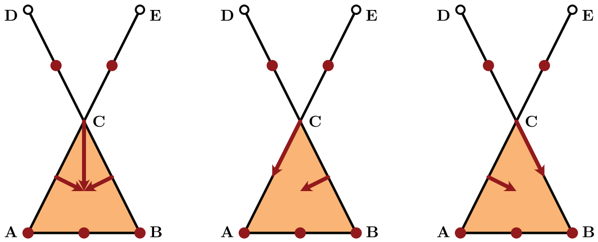
- The above result provides a categorical approach to connection matrix theory which allows for changes in the Morse decomposition poset.
- Connection matrices can be computed using the algorithms developed by Harker, Mischaikow, Spendlove (2018) and Dey, Lipinski, Mrozek, Slechta (2022).



	C_1	C_2	C_3
C_1		0	
C_2			1
C_3			

Multivectors and Nonunique Connection Matrices

Multivectors can be used in a natural way to allow for flow ambiguities. By breaking up the multivector and therefore choosing specific dynamical behavior, one can determine multiple connection matrices, which can be indicative of potential saddle-saddle connections.



d'	A	B	AB	CD	CE
A			-1	-1	-1
B			1		
AB					
CD					
CE					

d''	A	B	AB	CD	CE
A			-1		
B			1	-1	-1
AB					
CD					
CE					

Towards a General Theory

Kubica, Lipinski, Mrozek, W. (2019):

- Based on earlier work by Mrozek (2017), one can develop a complete Conley-Morse theory for combinatorial multivector fields on finite T_0 topological spaces.
- Multivectors seem to allow for easier passage from classical dynamical systems to discretized structures, as they can be used to avoid decisions of precise flow directions.

Barmak, Mrozek, W. (2020):

- One can prove a Lefschetz fixed point theorem for multivalued maps of finite T_0 topological spaces.

Barmak, Mrozek, W. (2022):

- In the discrete-time setting, it is possible to develop a Conley index theory for iterated multivalued maps on finite T_0 topological spaces.

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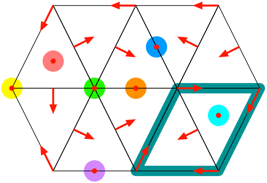
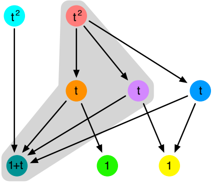
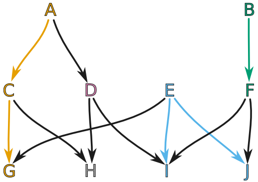
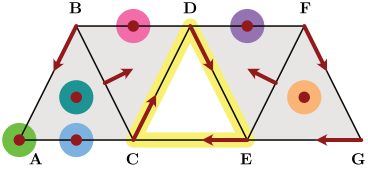
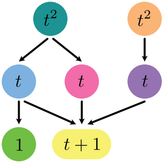
Thank You!











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